8

Planar #CSP for Symmetric Constraints

In Chapter 3 we proved a complexity dichotomy for #CSP($\mathcal{F}$), Theorem 3.7, where $\mathcal{F}$ is a set of constraint functions mapping Boolean inputs to complex numbers. The conclusion of Theorem 3.7 is that the problem #CSP($\mathcal{F}$) is #P-hard, unless $\mathscr{F} \subseteq \mathscr{A}$ or $\mathscr{F} \subseteq \mathscr{P}$, in which case the problem is computable in polynomial time. In this chapter, we would like to consider this problem restricted to planar input instances, i.e., the bipartite graph describing the connection between variables and constraints is a planar graph. Here we consider only symmetric constraint functions. We will show that in addition to $\mathscr{A}$ and $\mathscr{P}$, there is a third tractable class $\widehat{\mathscr{M}}$, which is the class of matchgate signatures $\mathscr{M}$ under a holographic transformation by the Hadamard matrix $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Everything else remains #P-hard.

8.1 Introduction

In terms of Holant problems, the problem #CSP($\mathcal{F}$) is equivalent to Holant($\mathcal{F} \cup \mathcal{EQ}$), where $\mathcal{EQ} = \{=_1, =_2, =_3, \dots\}$ and $(=_k) = [1, 0, \dots, 0, 1]$ is the equality signature of arity k. This equivalence is also valid between planar #CSP and planar Holant problems as well. Considering the edge–vertex incidence graph, one can see that Holant($\mathcal{F}$) is equivalent to Holant($=_2 \mid \mathcal{F}$). Then we can apply a holographic transformation, and see that for any nonsingular 2×2 matrix T, Holant($=_2 \mid \mathcal{F}$) is equivalent to Holant($(=_2)(T^{-1})^{\otimes 2} \mid T\mathcal{F}$). Here $T\mathcal{F} = \{T^{\otimes \operatorname{arity}(f)} f \mid f \in \mathcal{F}\}$.

Using the operator $\operatorname{Sym}_n^{n-1}$ and the expression of a sum of tensor powers, Theorem 4.11 can be equivalently stated as

Theorem 8.1. *Let f be a symmetric signature in $\mathscr{M}$ of arity n. Then there exist $a, b, \lambda \in \mathbb{C}$ such that f takes one of the following forms:*

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1. $[a,b]^{\otimes n}+[a,-b]^{\otimes n}=\begin{cases}2[a^n,0,a^{n-2}b^2,0,\ldots,0,b^n] & n \text{ is even,}\\ 2[a^n,0,a^{n-2}b^2,0,\ldots,0,ab^{n-1},0] & n \text{ is odd;}\end{cases}$
2. $[a,b]^{\otimes n}-[a,-b]^{\otimes n}=\begin{cases}2[0,a^{n-1}b,0,a^{n-3}b^3,0,\ldots,0,ab^{n-1},0] & n \text{ is even,}\\ 2[0,a^{n-1}b,0,a^{n-3}b^3,0,\ldots,0,b^n] & n \text{ is odd;}\end{cases}$
3. $\lambda\,\mathrm{Sym}_n^{n-1}([1,0];[0,1])=[0,\lambda,0,\ldots,0]$;
4. $\lambda\,\mathrm{Sym}_n^{n-1}([0,1];[1,0])=[0,\ldots,0,\lambda,0]$.

In (4.23) and (4.24) of Chapter 4 these expressions are proved to be equivalent to the list in Theorem 4.11.

From the Holant perspective, the signatures in $\mathcal{EQ}$ are always available in #CSP($\mathcal{F}$). By the signature theory of matchgates and their transformations, the Hadamard matrix H_2 is essentially the only holographic transformation under which all equalities in $\mathcal{EQ}$ become realizable as standard signatures of matchgates $\mathcal{M}$. To see this, recall the definition of the equivalence relation $\sim$ given in Proposition 4.24, which defines the basis manifold $\mathcal{B}=\mathrm{GL}_2(\mathbb{C})/\sim$. Then in Lemma 4.29, we take $\lambda=\mu=1$ and $\alpha=0$ to get the set of all holographic transformations T such that $(=_k)(T^{-1})^{\otimes k}\in\mathcal{M}$. As representatives in $\mathcal{B}$, these T take the form $T=\begin{bmatrix}1 & \omega\\ 1 & -\omega\end{bmatrix}$, where $\omega^{2k}=1$. Since $\mathcal{EQ}$ contains $(=_k)$ for all $k\geq 1$, we have $\omega=\pm 1$, i.e., the only common transformations T represented in $\mathcal{B}$ are $H_2=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ or $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}H_2=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$. By the definition of $\mathcal{B}$, this means that T has the form $\begin{bmatrix}x & 0\\ 0 & y\end{bmatrix}H_2$ or $\begin{bmatrix}0 & x\\ y & 0\end{bmatrix}H_2$, for $xy\neq 0$. However, for any recognizer signature Γ of arity n written as a row vector, $\Gamma\in\mathcal{M}$ iff $\Gamma\begin{bmatrix}x & 0\\ 0 & y\end{bmatrix}^{\otimes n}\in\mathcal{M}$, and also iff $\Gamma\begin{bmatrix}0 & x\\ y & 0\end{bmatrix}^{\otimes n}\in\mathcal{M}$. Equivalently, if Γ is a generator signature of arity n written as a column vector, $\Gamma\in\mathcal{M}$ iff $\begin{bmatrix}1/x & 0\\ 0 & 1/y\end{bmatrix}^{\otimes n}\Gamma\in\mathcal{M}$, and also iff $\begin{bmatrix}0 & 1/y\\ 1/x & 0\end{bmatrix}^{\otimes n}\Gamma\in\mathcal{M}$. This can be seen by a combinatorial construction, as well as by MGI. For symmetric signatures this is also obvious by Theorem 8.1. Thus, given Holant($\mathcal{EQ}\mid\mathcal{F}$), to test whether there is any holographic transformation T such that $(\mathcal{EQ})T^{-1}\subseteq\mathcal{M}$ and $T\mathcal{F}\subseteq\mathcal{M}$, we may consider H_2 as the only possible transformation, namely whether $H_2\mathcal{F}\subseteq\mathcal{M}$.

Let $\widehat{\mathcal{F}}$ denote $H_2\mathcal{F}$ for any set $\mathcal{F}$ of signatures. Note that H_2^{-1} is H_2 up to a nonzero constant, hence $\widehat{\widehat{\mathcal{F}}}=\mathcal{F}$. It is easy to see that $\widehat{\mathcal{EQ}}$ is $\{[1,0],[1,0,1],[1,0,1,0],\ldots\}$. Therefore, #CSP($\mathcal{F}$) $\equiv_{\mathrm{T}}$ Holant($\mathcal{EQ}\mid\mathcal{F}$) $\equiv_{\mathrm{T}}$ Holant($\mathcal{F}\cup\mathcal{EQ}$), is equivalent to Holant($\widehat{\mathcal{F}}\cup\widehat{\mathcal{EQ}}$). The equivalence

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of #CSP($\mathcal{F}$), Holant($\mathcal{F} \cup \mathcal{EQ}$), and Holant($\widehat{\mathcal{F}} \cup \widehat{\mathcal{EQ}}$) is also valid over planar instances. We denote by Pl-#CSP($\mathcal{F}$), Pl-Holant($\mathcal{F} \cup \mathcal{EQ}$), and Pl-Holant($\widehat{\mathcal{F}} \cup \widehat{\mathcal{EQ}}$), the planar versions of the corresponding problems #CSP($\mathcal{F}$), Holant($\mathcal{F} \cup \mathcal{EQ}$), and Holant($\widehat{\mathcal{F}} \cup \widehat{\mathcal{EQ}}$) respectively.

The main dichotomy theorem of this chapter is due to Guo and Williams [GW13]. With their permission, we will closely follow their presentation. This dichotomy states that, for #CSP problems over any set of complex-valued *symmetric* constraint functions on Boolean variables, Valiant's holographic transformation followed by Kasteleyn's algorithm (Theorem 4.7) is a *universal* methodology to solve in polynomial time on planar graphs those counting problems that are #P-hard in general. This theorem generalizes a previous theorem with the same conclusion for real-valued symmetric signatures [CLX10].

Theorem 8.2 (Guo-Williams)**.** *Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then* Pl-#CSP($\mathcal{F}$) *is* #P*-hard unless $\mathcal{F}$ satisfies one of the following conditions, in which case it is tractable:*

1. #CSP($\mathcal{F}$) *is tractable, i.e., $\mathcal{F} \subseteq \mathcal{A}$ or $\mathcal{F} \subseteq \mathcal{P}$ (by Theorem 3.7).*
2. *$\widehat{\mathcal{F}}$ is realizable as standard matchgate signatures, i.e., $\mathcal{F} \subseteq \widehat{\mathcal{M}}$.*

Thus, for symmetric complex-valued signatures in Boolean variables, to be transformable to matchgates under the Hadamard transformation followed by Kasteleyn's algorithm (a.k.a. the FKT algorithm; see Chapter 4) is a *universal* methodology to solve Pl-#CSP problems in polynomial time over planar graphs that are #P-hard over general graphs.

We note that Theorem 8.2 is proved only for sets of *symmetric* signatures, whereas Theorem 3.7 does not have this restriction. In Chapter 10 we will state the dichotomy theorem for Pl-#CSP($\mathcal{F}$), Theorem 10.1, which is valid for general asymmetric signatures, i.e., without the restriction that the signatures be symmetric. We will not give the full proof of Theorem 10.1, but a main engine of the proof is to use Theorem 8.2. We also note that the tractability criteria of Theorem 8.2 are decidable in polynomial time in the length of the description of the symmetric signatures in $\mathcal{F}$. This follows from the forms of symmetric signatures $\mathcal{F}_{123}$ in $\mathcal{A}$, and those in $\mathcal{P}$ and in $\widehat{\mathcal{M}}$. Note that for $f = [a, b]^{\otimes n} \pm [a, -b]^{\otimes n} \in \mathcal{M}$, when written as a column vector, its transformation $H_2 f \in \widehat{\mathcal{M}}$ has the form $[x, y]^{\otimes n} \pm [y, x]^{\otimes n}$, where $x = a + b$ and $y = a - b$. (Recall that we write a row vector and its column vector transpose interchangeably, provided it is suitable for matrix-vector product.)

In proving dichotomy theorems for Boolean #CSP($\mathcal{F}$), pinning is an important tool. If one can realize the unary pinning signatures Is-Zero $\Delta_0 = [1, 0]$

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and Is-One $\Delta_1 = [0, 1]$, then one can obtain all subsignatures of any $f \in \mathcal{F}$. However, the proof in Chapter 3 for #CSP($\mathcal{F}$) obtaining Δ_0 and Δ_1 is by a *nonplanar* reduction. More intrinsically, we note that in the nonplanar setting, Δ_0 and Δ_1 are contained in both maximal tractable sets $\mathscr{A}$ and $\mathscr{P}$. Therefore, it is plausible to prove the pinning reduction #CSP($\mathcal{F} \cup \{\Delta_0, \Delta_1\}$) $\leq_T$ #CSP($\mathcal{F}$), as was done in Lemma 3.13; pinning in this setting would not imply the collapse of #P to P. However, $\mathcal{EQ}$ and $\{[1, 0], [0, 1]\}$ are not simultaneously realizable as matchgates, under any holographic transformation. Suppose Theorem 8.2 is true. Then we should not expect to be able to pin for Pl-#CSP($\mathcal{F}$), i.e., to prove a general pinning reduction Pl-#CSP($\mathcal{F} \cup \{\Delta_0, \Delta_1\}$) $\leq_T$ Pl-#CSP($\mathcal{F}$), since otherwise #P collapses to P. The collapse is seen by taking $\mathcal{F} = \widehat{\mathscr{M}}$, which is neither a subset of $\mathscr{A}$ nor $\mathscr{P}$, and $\Delta_0 \notin \widehat{\mathscr{M}}$ since $H_2\Delta_0 = [1, 1] \notin \mathscr{M}$.

So we cannot expect to pin in the *primal* setting Pl-#CSP($\mathcal{F}$) in general. Instead, we apply the Hadamard transformation and consider in the *dual* setting Pl-Holant($\widehat{\mathcal{F}} \cup \widehat{\mathcal{EQ}}$). In this Hadamard basis, it is at least possible again that we can prove a pinning reduction for planar Holant problems, since Δ_0 and Δ_1 are contained in $\mathscr{A} \cap \widehat{\mathscr{P}} \cap \mathscr{M}$. Note that $\widehat{\mathscr{A}} = \mathscr{A}$ and $\widehat{\mathscr{P}}$ still contains all unary signatures. If $\mathcal{F} \subseteq \mathscr{A}$, or $\mathscr{P}$, or $\widehat{\mathscr{M}}$, then $\widehat{\mathcal{F}} \cup \widehat{\mathcal{EQ}} \cup \{\Delta_0, \Delta_1\} \subseteq \mathscr{A}$, or $\widehat{\mathscr{P}}$ or $\mathscr{M}$ respectively. Indeed, we will prove our pinning result in this Hadamard basis in Section 8.6, see (8.5).

For convenience we will denote Pl-Holant($\mathcal{F} \cup \{\Delta_0, \Delta_1\}$) by Pl-Holant^c($\mathcal{F}$).

Our plan for the proof of Theorem 8.2 is as follows. Instead of proving in the primal setting, we will prove the theorem mainly in the dual setting Pl-Holant($\widehat{\mathcal{F}} \cup \widehat{\mathcal{EQ}}$). For Holant problems, as in #CSP problems, it is often important to understand the complexity of the small arity cases first. We already proved Theorem 6.3, a dichotomy for Pl-Holant(f), where f is a symmetric signature of arity 3. After some strengthening of unary interpolations, we will prove a dichotomy for Pl-Holant(f), where f is a symmetric signature of arity 4. This uses the idea of a *planar pairing*. If every signature in $\mathcal{F}$ has even arity, then by a simple parity argument, every $\mathcal{F}$-gate will also have even arity. Thus it is impossible to construct directly a gadget with an odd arity. After we prove some useful lemmas using a technique called *domain pairing* we will circumvent this difficulty and *functionally* realize an odd arity signature using only signatures of even arity. After that we show some no-mixing lemmas, which say that the known classes of tractable signatures for Pl-#CSP become #P-hard when mixed. Then we realize pinning in the Hadamard basis. After that we prove the Pl-#CSP dichotomy for a single signature. The main dichotomy for Pl-#CSP($\mathcal{F}$), Theorem 8.2, follows easily from this and the no-mixing lemmas.

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For easy reference we list some results in a convenient form from previous chapters. Note that the problem Pl-Holant($[x, y, z] \mid \mathcal{EQ}$) is just Pl-#CSP($[x, y, z]$), which is also equivalent to a planar spin system with edge function $[x, y, z]$.

Theorem 8.3. *For any $x, y, z \in \mathbb{C}$, both* Pl-Holant($[x, y, z] \mid \mathcal{EQ}$) *and* Pl-Holant($[x, y, z] \mid \widehat{\mathcal{EQ}}$) *are* #P*-hard unless one of the following conditions holds, in which case both problems are computable in polynomial time:*

1. $xz = y^2$.
2. $y = 0$.
3. $xz = -y^2$ *and* $x = -z$.
4. $x = z$.

Proof. Theorem 5.1 (equivalently Theorem 5.3) was proved for the cases $k = 3$ and 4 in Chapter 5. For Pl-Holant($[x, y, z] \mid \mathcal{EQ}$) both $(=_3)$ and $(=_4)$ are present in $\mathcal{EQ}$, and thus Pl-Holant($[x, y, z] \mid \mathcal{EQ}$) is #P-hard unless both tractability criteria of Theorem 5.3 for $k = 3$ and $k = 4$ are satisfied. This gives the listed tractability criterion. Conversely, if one of the tractability conditions is satisfied then clearly Pl-Holant($[x, y, z] \mid \mathcal{EQ}$) is computable in P.

Under the transformation $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, it is easy to see that the conditions $xz = y^2$ and $(xz = -y^2) \wedge (x = -z)$ are invariant while the conditions $y = 0$ and $x = z$ map to each other. Therefore, by an apparent coincidence, the tractability conditions remain the same for Pl-Holant($[x, y, z] \mid \widehat{\mathcal{EQ}}$). □

Theorem 8.4. *Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then* Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is computable in polynomial time if $\mathcal{F} \subseteq \mathcal{A}$, $\mathcal{F} \subseteq \widehat{\mathcal{P}}$, or $\mathcal{F} \subseteq \mathcal{M}$.*

Proof. We only need to observe that $\widehat{\mathcal{EQ}} \cup \{[1, 0], [0, 1]\} \subset \mathcal{A} \cap \widehat{\mathcal{P}} \cap \mathcal{M}$. The case $\mathcal{F} \subseteq \mathcal{A}$ follows from Theorem 6.12. The case $\mathcal{F} \subseteq \widehat{\mathcal{P}}$ follows from Theorem 6.12 and Theorem 3.5. The case $\mathcal{F} \subseteq \mathcal{M}$ follows from Theorem 4.7. □

8.2 Unary Interpolation Revisited

In this section, we discuss *unary recursive construction* and give an exact condition for when it succeeds. The goal of this construction is to interpolate an arbitrary unary signature.

For a unary recursive construction in the Holant framework, there are two gadgets involved: a *starter* gadget of arity 1 and a *recursive* gadget of arity 2. In the Pl-Holant framework the gadgets must also be planar. The signature of the

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Figure 8.1. Unary recursive construction with starter gadget.

starter gadget is represented by a vector $s \in \mathbb{C}^2$. The signature of the recursive gadget is represented by a matrix $M \in \mathbb{C}^{2\times 2}$. The construction begins with the starter gadget and proceeds by connecting $k \geq 0$ recursive gadgets, one at a time, to the only available edge (see Figure 8.1). The signature matrix of the resulting gadget is $M^k s$. This construction is denoted by (M, s).

The most demanding part of the requirement in these constructions, for the purpose of polynomial interpolation, is to construct an infinite set of signatures that are pairwise linearly independent (see Chapter 5). The pairwise linear independence of signatures translates into distinct evaluation points for the polynomial being interpolated.

We say two vectors are orthogonal if their dot product is 0. We first prove a lemma that gives a determinantal condition for the statement that a vector s is not orthogonal to any row eigenvector of a matrix M.

Lemma 8.5. *Suppose $M \in \mathbb{C}^{n\times n}$ and $s \in \mathbb{C}^n$. Then s is not orthogonal to any row eigenvector of M iff* $\det([s\, Ms \ldots M^{n-1}s]) \neq 0$.

Proof. Suppose $\det([s\, Ms \ldots M^{n-1}s]) = 0$. Then there is a nonzero row vector v such that $v[s\, Ms \ldots M^{n-1}s] = 0$ is the zero vector. Consider the linear span S by row vectors in the set $\{v, vM, \ldots, vM^{n-1}\}$. By the Cayley–Hamilton theorem, M satisfies its own characteristic polynomial, which is a monic polynomial of degree n. Thus, M^n is a linear combination of $I_n, M, \ldots, M^{n-1}$. This shows that S is an invariant subspace of row vectors under the action of multiplication by M from the right. Being an invariant subspace over $\mathbb{C}$, there exists a row eigenvector $u \in S$ of M, and the dot product of s with u is 0.

Conversely, suppose $\det([s\, Ms \ldots M^{n-1}s]) \neq 0$ and assume for a contradiction that s is orthogonal to some row eigenvector v of M with eigenvalue λ. Then $v[s\, Ms \ldots M^{n-1}s] = 0$ is the zero vector because $vM^i s = \lambda^i vs = 0$. Since $v \neq 0$, this is a contradiction. □

To construct an infinite set of signatures that are pairwise linearly independent, a necessary condition is that M has infinite order modulo a scalar. Otherwise, $M^k = cI_n$ for some $k > 0$ and $c \neq 0$, then any vector of the form $M^\ell s$ for $\ell \geq k$ is a multiple of a vector in the set $\{M^i s\}_{0\leq i<k}$.

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Lemma 8.6. *Suppose $M \in \mathbb{C}^{n\times n}$ and $s \in \mathbb{C}^n$. If the following three conditions are satisfied,*

1. $\det(M) \neq 0$.
2. $\det([s\ Ms \ldots M^{n-1}s]) \neq 0$.
3. *M has infinite order modulo a scalar.*

then the vectors in the set $V = \{M^k s\}_{k\geq 0}$ are pairwise linearly independent.

Proof. Since $\det(M) \neq 0$, M is nonsingular and all eigenvalues are nonzero. Let $M = P^{-1}JP$ be the Jordan decomposition of M and let $v = Ps \in \mathbb{C}^n$. Clearly $v \neq 0$ since $s \neq 0$ by condition 2. Suppose for a contradiction that the vectors in V are not pairwise linearly independent. This means that there exist integers $k > \ell \geq 0$ such that $M^k s = cM^\ell s$ for some nonzero $c \in \mathbb{C}$. Let $m = k - \ell > 0$. Then we have $P^{-1}J^m Ps = M^m s = cs$ and thus $J^m v = cv$.

Suppose that J contains some nontrivial Jordan block and consider the 2×2 submatrix $\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}$ in the bottom right corner of this block, where λ is an eigenvalue of M. Without loss of generality we may assume that the rows and colmuns of this 2×2 submatrix are indexed by $n-1$ and n. From this portion of J, the two equations given by $J^m v = cv$ are $\lambda^m v_{n-1} + m\lambda^{m-1} v_n = cv_{n-1}$ and $\lambda^m v_n = cv_n$. By Lemma 8.5, s is not orthogonal to any row eigenvector of M; therefore $v_n \neq 0$ since the last row of P is a row eigenvector of M. But then these equations imply that $m\lambda^{m-1} v_n = 0$, a contradiction.

Otherwise, J is a diagonal matrix $\operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$. From $J^m v = cv$, we get the equations $\lambda_i^m v_i = cv_i$ for $1 \leq i \leq n$. As $PM = JP$ the rows of P are row eigenvectors of M. By definition $v = Ps$, and since s is not orthogonal to any row eigenvector of M, $v_i \neq 0$ for $1 \leq i \leq n$. But then $M^m = cI_n$, which contradicts the assumption that M has infinite order modulo a scalar. □

These conditions guarantee success of interpolation by a unary recursive construction.

Lemma 8.7. *Let $\mathcal{F}$ be a set of signatures. If there exist a planar $\mathcal{F}$-gate with signature matrix $M \in \mathbb{C}^{2\times 2}$ and a planar $\mathcal{F}$-gate with signature $s \in \mathbb{C}^2$ satisfying the following conditions:*

1. $\det(M) \neq 0$;
2. $\det([s\ Ms]) \neq 0$;
3. *M has infinite order modulo a scalar;*

then Pl-Holant$(\mathcal{F} \cup \{[a, b]\}) \leq_{\mathrm{T}}$ Pl-Holant$(\mathcal{F})$ *for any $a, b \in \mathbb{C}$.*

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Proof. Consider an instance Ω of Pl-Holant($\mathcal{F} \cup \{[a, b]\}$). Let V' be the subset of vertices assigned $[a, b]$ and suppose that $|V'| = n$. We construct from Ω a sequence of instances Ω_k of Pl-Holant($\mathcal{F}$) indexed by $k \geq 0$. We obtain Ω_k from Ω by replacing each occurrence of $[a, b]$ with the unary recursive construction (M, s) in Figure 8.1 containing k copies of the recursive gadget. This unary recursive construction has the signature $[x_k, y_k] = M^k s$.

By Lemma 8.6, the signatures in the set $V = \{[x_k, y_k] | 0 \leq k \leq n + 1\}$ are pairwise linearly independent. In particular, at most one y_k can be 0.

We stratify the assignments in Ω based on the assignment to $[a, b]$. Let c_ℓ be the sum over all assignments of products of evaluations at all vertices other than those from V' such that exactly ℓ occurrences of $[a, b]$ have their incident edge assigned 0 (and $n - \ell$ have their incident edge assigned 1). Then

$$\text{Holant}(\Omega) = \sum_{0 \leq \ell \leq n} a^\ell b^{n-\ell} c_\ell$$

and the value of the Holant on Ω_k, for $0 \leq k \leq n + 1$, is $\text{Holant}(\Omega_k) = \sum_{0 \leq \ell \leq n} x_k^\ell y_k^{n-\ell} c_\ell$, and for at least $n + 1$ of them where $y_k \neq 0$, we have

$$\frac{\text{Holant}(\Omega_k)}{y_k^n} = \sum_{0 \leq \ell \leq n} \left(\frac{x_k}{y_k}\right)^\ell c_\ell.$$

This is a linear system with unknowns c_ℓ and a Vandermonde coefficient matrix. Since the signatures in V are pairwise linearly independent, the ratios x_k/y_k are all distinct (and well-defined since $y_k \neq 0$), which implies that the Vandermonde matrix has full rank. Therefore, we can solve the linear system for the unknown c_ℓ's and obtain the value of Holant(Ω). □

With this lemma, we obtain a tight characterization for the success of interpolation by a unary recursive construction. For example, the construction using a recursive gadget with signature matrix $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and a starter gadget with signature $s = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is successful because M and s satisfy the conditions of Lemma 8.6 even though the eigenvalues of M are equal.

The first two conditions of Lemma 8.7 are easy to check. The third condition holds in one of these two cases: either the eigenvalues are the same but M is not a multiple of the identity matrix, or the eigenvalues are different but their ratio is not a root of unity.

These conditions work well with the anti-gadget technique (see Chapter 5). The power of this lemma is that when the third condition fails to hold, there

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exists an integer m such that $M^m = cI_2$, where $c \neq 0$ and I_2 is the 2×2 identity matrix. Therefore we can construct $M^{m-1} = cM^{-1}$, which is effectively the same as M^{-1}, and use this in other gadget constructions. We remark that if any such m exists then the minimum such m is bounded by a constant. Regardless of the number of internal nodes of the $\mathcal{F}$-gate, its signature entries in $M \in \mathbb{C}^{2\times 2}$ all belong to the fixed algebraic extension field $\mathbf{F}$ over $\mathbb{Q}$, from which functions in $\mathcal{F}$ take values. The ratio of the two eigenvalues λ and μ of M belongs to the splitting field which has a degree of extension at most $2[\mathbf{F} : \mathbb{Q}]$. If M has finite order modulo a scalar, then let m be the minimum exponent $\ell > 0$ such that $M^\ell = cI_2$ for some $c \neq 0$. Then $m = \min\{\ell > 0 \mid \left(\frac{\lambda}{\mu}\right)^\ell = 1\}$. Thus the mth primitive root of unity $e^{2\pi \mathrm{i}/m}$ belongs to the splitting field. However the mth cyclotomic field $\Phi_m = \mathbb{Q}(e^{2\pi \mathrm{i}/m})$ has degree of extension $[\Phi_m : \mathbb{Q}] = \phi(m)$ over the rational numbers $\mathbb{Q}$, where $\phi(m)$ is the Euler's totient function. By an elementary estimate $\phi(m)^2 \geq m/2$. Hence m is upper bounded by a constant only depending on $\mathbf{F}$.

8.3 Planar Pairing

In this section, we prove a dichotomy for Pl-Holant(f), where f is a symmetric signature of arity 4. The key part is to consider the problem Pl-Holant($[t, 1, 0, 0, 0]$) when t is different from 0. Over the next two lemmas, we prove that this problem is #P-hard by reducing from Pl-Holant($[t, 1, 0, 0]$). These problems are weighted versions of counting matchings over planar k-regular graphs for $k = 4$ and $k = 3$ respectively. We use the idea of a planar pairing to be defined in Definition 8.9, and apply Lemma 5.31.

In Lemma 8.8, we show how to use either the anti-gadget technique or the unary interpolation technique based on Lemma 8.7 from Section 8.2 to effectively obtain $[1, 0, 0]$. The construction in this proof is actually not a unary recursive construction, but a binary recursive construction. However, the particular forms in the starter and recursive gadgets permit analysis equivalent to that of a unary recursive construction.

Lemma 8.8. *For any $t \in \mathbb{C}$ and signature set $\mathcal{F}$ containing $[t, 1, 0, 0, 0]$,*

$$\text{Pl-Holant}(\mathcal{F} \cup \{[1, 0, 0]\}) \leq_{\mathrm{T}} \text{Pl-Holant}(\mathcal{F}).$$

Proof. Consider the gadget construction in Figure 8.2. For $k \geq 0$, the signature of N_k is of the form $[a_k, b_k, 0]$, and $N_0 = [t, 1, 0]$. Since N_k is symmetric and always ends with 0, we can analyze this construction as though it were a unary

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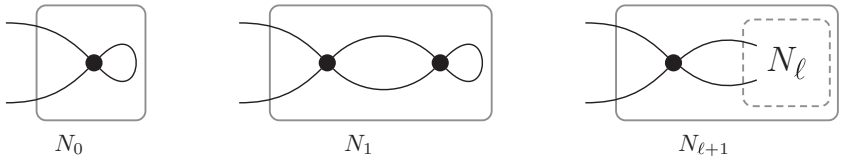

Figure 8.2. Binary recursive construction with starter gadget used to interpolate $[1, 0, 0]$. The vertices are assigned $[t, 1, 0, 0, 0]$.

recursive construction. Let $s_k = \begin{bmatrix} a_k \\ b_k \end{bmatrix}$, so $s_0 = \begin{bmatrix} t \\ 1 \end{bmatrix}$. It is clear that $s_k = M^k s_0$, where $M = \begin{bmatrix} t & 2 \\ 1 & 0 \end{bmatrix}$.

Now M is nonsingular since $\det(M) = -2$. If M has finite order modulo a scalar, then $M^m = cI_2$ for some integer $m \geq 1$ and some nonzero $c \in \mathbb{C}$. Then N_{m-1} is an anti-gadget of M with signature $M^{m-1}s_0 = cM^{-1}s_0 = c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ here denotes the signature $[1, 0, 0]$ as an abbreviated notation. Hence we have $[1, 0, 0]$ after normalizing.

Now assume that M has infinite order modulo a scalar. Since $\det([s_0\, Ms_0]) = -2$, we can interpolate any signature of the form $[x, y, 0]$ by adapting Lemma 8.7, including $[1, 0, 0]$. □

Note that having $[1, 0, 0] = [1, 0]^{\otimes 2}$ is equivalent to be able to pin *two variables* to 0 simultaneously. To apply $[1, 0]^{\otimes 2}$ in a planar setting, one must be able to connect the two variables in a planar way.

Definition 8.9 (Planar pairing)**.** A *planar pairing* M in a planar graph $G = (V, E)$ is a partition of V into pairs such that $(V, E \cup M)$ is a planar multigraph, where $E \cup M$ is the multiset formed by disjoint union.

Lemma 5.31 shows that for any 3-regular planar graph, a planar pairing exists and can be easily found in polynomial time. Now we use the planar pairing technique to show the following.

Lemma 8.10. *Let* $t \in \mathbb{C}$. *Then* Pl-Holant$([t, 1, 0, 0])$ $\leq_{\mathrm{T}}$ Pl-Holant$([t, 1, 0, 0, 0, 0])$.

Proof. An instance of Pl-Holant$([t, 1, 0, 0])$ is a signature grid Ω with underlying graph $G = (V, E)$ that is planar and 3-regular. By Lemma 5.31, there exists a planar pairing M in G and it can be found in polynomial time. Then $G' = (V, E \cup M)$ is a planar and 4-regular multigraph. We assign $[t, 1, 0, 0, 0]$

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to every vertex in G'. By Lemma 8.8, we can assume that we have $[1, 0, 0]$. We replace each edge in M with a path of length 2 to form a graph G'' and assign $[1, 0, 0] = [1, 0]^{\otimes 2}$ to the new vertices on the paths. Note that applying $[1, 0]$ to $[t, 1, 0, 0, 0]$ gives $[t, 1, 0, 0]$. Then the signature grid Ω'' with underlying graph G'' has the same Holant value as the original signature grid Ω. □

Corollary 8.11. *The problem* Pl-Holant($[t, 1, 0, 0, 0]$) *is* #P*-hard for all nonzero* $t \in \mathbb{C}$.

Proof. By Theorem 6.3, the planar Holant dichotomy for a single ternary signature, the problem Pl-Holant($[t, 1, 0, 0]$) is #P-hard. Then Corollary 8.11 follows from Lemma 8.10. □

One feature of this reduction is worth noticing. Note that from any set of signatures all with even arities, any gadget construction will produce a signature also of even arity. This is by a simple parity argument, because the sum of arities of all signatures used in the construction is twice the number of internal edges plus the arity of the signature produced. Thus, with $[t, 1, 0, 0, 0]$ alone one cannot construct any signature of arity 3 by gadget construction.

Now we are ready to prove our Pl-Holant dichotomy for a symmetric arity 4 signature.

Theorem 8.12. *If* f *is a nondegenerate, symmetric, complex-valued signature of arity* 4 *in Boolean variables, then* Pl-Holant(f) *is* #P*-hard unless* f *is* $\mathscr{A}$*-transformable or* $\mathscr{P}$*-transformable or vanishing or* $\mathscr{M}$*-transformable, in which case the problem is computable in polynomial time.*

Proof. Let $f = [f_0, f_1, f_2, f_3, f_4]$. Then its compressed matrix

$$\widetilde{M}_f = \begin{bmatrix} f_0 & 2f_1 & f_2 \\ f_1 & 2f_2 & f_3 \\ f_2 & 2f_3 & f_4 \end{bmatrix}.$$

If $\det(\widetilde{M}_f) \neq 0$, equivalently if there do not exist $a, b, c \in \mathbb{C}$, not all zero, such that for all $k \in \{0, 1, 2\}$, $af_k + bf_{k+1} + cf_{k+2} = 0$, then Pl-Holant($f$) is #P-hard by Corollary 7.25. Otherwise, there do exist such a, b, c. If $a = c = 0$, then $b \neq 0$, so $f_1 = f_2 = f_3 = 0$. In this case, $f \in \mathscr{P}$ is a generalized equality signature, so f is $\mathscr{P}$-transformable. Now suppose a and c are not both 0. If $b^2 - 4ac \neq 0$, then $f_k = \alpha^{4-k}\beta^k + \gamma^{4-k}\delta^k$, where $\det \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \neq 0$. We apply a holographic transformation by $M = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$ to Pl-Holant(f) $\equiv_{\mathrm{T}}$ Pl-Holant($=_2 | f$). M transforms f to $(M^{-1})^{\otimes 4} f = (=_4)$, and the left-hand side $(=_2)$ to a

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binary symmetric signature $g = (=_2)M^{\otimes 2}$. Then we apply Theorem 5.1 (the case $k = 4$ is proved in Chapter 5) to show that Pl-Holant(f) is #P-hard unless f is either $\mathscr{A}$-, $\mathscr{P}$-, or $\mathscr{M}$-transformable. Otherwise, $b^2 - 4ac = 0$ and there are two cases. In the first, for any $0 \le k \le 4$, $f_k = ck\lambda^{k-1} + d\lambda^k$, where $c \neq 0$. In the second, for any $0 \le k \le 4$, $f_k = c(4-k)\lambda^{3-k} + d\lambda^{4-k}$, where $c \neq 0$. These cases map between each other under a holographic transformation by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so assume that we are in the first case. If $\lambda = \pm \mathrm{i}$, then f is vanishing. Otherwise, a further holographic transformation by $\frac{1}{\sqrt{1+\lambda^2}} \begin{bmatrix} 1 & \lambda \\ \lambda & -1 \end{bmatrix}$ transforms f to $\hat{f} = [t, 1, 0, 0, 0]$ for some $t \in \mathbb{C}$ after normalizing the second entry. If $t = 0$, then the problem is counting perfect matchings over planar 4-regular graphs, so $\hat{f} \in \mathscr{M}$ and f is $\mathscr{M}$-transformable. Otherwise, $t \neq 0$ and by Corollary 8.11, we are done. □

8.4 Domain Pairing

In this section, we discuss a technique called domain pairing, which pairs input variables to simulate a problem on a domain of size 4 and then reduces a problem in the Boolean domain to it. As explained in Section 8.1, we work in the Hadamard basis instead of the standard basis. The goal therefore is a dichotomy for Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$).

By a simple parity argument, gadgets constructed with signatures of even arity can only realize other signatures of even arity. In particular, this means that $(=_4)$ cannot by itself be used to construct $(=_3)$. Nevertheless, there is an indirect method that can realize $(=_3)$ using $(=_4)$. The idea is to change the domain from individual elements to pairs of elements. Thus, we call this reduction technique *domain pairing*.

The following is a lemma in which the reduction is carried out by this method.

Lemma 8.13 (Domain pairing). *Let $a, b, x, y \in \mathbb{C}$. Suppose $f = [x, 0, y, 0]$ and $g = [a, 0, \ldots, 0, b]$ with arity ≥ 3. If $aby \neq 0$ and $x^2 \neq y^2$, then*

$$\text{Pl-Holant}([x, y, y] \mid \mathcal{EQ}) \le_{\mathrm{T}} \text{Pl-Holant}(\{f, g\} \cup \widehat{\mathcal{EQ}}).$$

In particular, Pl-Holant($\{f, g\} \cup \widehat{\mathcal{EQ}}$) *is* #P*-hard.*

Proof. The problem Pl-Holant($[x, y, y] \mid \mathcal{EQ}$) is #P-hard by Theorem 8.3 since $y \neq 0$ and $x^2 \neq y^2$. Hence we only need to prove the reduction. Since we have $g = [a, 0, \ldots, 0, b]$ in Pl-Holant($\{f, g\} \cup \widehat{\mathcal{EQ}}$), by Lemma 7.48, we have $\mathcal{EQ}_2$, the set of all EQUALITIES of even arity.

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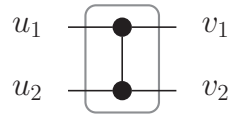

Figure 8.3. Gadget designed for the paired domain. One vertex is assigned $[1, 0, 1, 0]$ and the other is assigned $[x, 0, y, 0]$.

We remark that if one were able to obtain $(=_3)$ on the Boolean domain, then one could have constructed $[x, y, y]$ from $[x, 0, y, 0]$ easily. Simply taking a self-loop on $(=_3)$ gives $[1, 1]$, and connecting $[1, 1]$ to $[x, 0, y, 0]$ gives $[x, y, y]$. However, as noted earlier, there is an obstacle to obtaining $(=_3)$ due to parity. Instead we use the domain pairing technique.

An instance of Pl-Holant$([x, y, y] \mid \mathcal{EQ})$ is a planar bipartite graph $G = (U, V, E)$ in which every vertex in U has degree 2. We replace every vertex in V of degree k (which is assigned $(=_k) \in \mathcal{EQ}$) with a vertex of degree $2k$ and assign $(=_{2k}) \in \mathcal{EQ}_2$. Furthermore, we bundle two adjacent edges (representing variables) to form k bundles of two edges each. The k bundles correspond to the k incident edges of the original vertex with degree k.

If the inputs to these EQUALITY signatures are restricted to $\{(0, 0), (1, 1)\}$ on each bundle, then these EQUALITY signatures take value 1 on $((0, 0), \ldots, (0, 0))$ and $((1, 1), \ldots, (1, 1))$ and take value 0 elsewhere. Thus, if we restrict the domain to $\{(0, 0), (1, 1)\}$, it is the EQUALITY signature $(=_k)$.

To simulate $[x, y, y]$, we connect $f = [x, 0, y, 0]$ to $e = [1, 0, 1, 0] \in \widehat{\mathcal{EQ}}$ by a single edge as shown in Figure 8.3 to form a gadget with signature

$$h(u_1, u_2, v_1, v_2) = \sum_{w=0,1} f(u_1, v_1, w) \cdot e(u_2, v_2, w).$$

We replace every (degree 2) vertex in U (which is assigned $[x, y, y]$) by a degree 4 vertex assigned h, where the variables of h are bundled as (u_1, u_2) and (v_1, v_2).

The vertices in this new graph G' are connected as in the original graph G, except that every original edge is replaced by two edges that connect to the same side of the gadget in Figure 8.3, either the left or the right side. Clearly G' is still a planar graph. Notice that h is only connected by (u_1, u_2) and (v_1, v_2) to some bundle of two incident edges of an EQUALITY signature. Since this EQUALITY signature enforces that the value on each bundle is either $(0, 0)$ or $(1, 1)$, we need only to consider the restriction of h to the domain $\{(0, 0), (1, 1)\}$. On this domain, h is exactly the *symmetric* binary signature $[x, y, y]$. Therefore, the Holant of G' has the same Holant value as the original graph G. □

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Lemma 8.13 is mainly used to prove that generally speaking, Gen-Eq that do not belong to $\mathscr{A}$ cannot mix with $\widehat{\mathcal{EQ}}$. As $\widehat{\mathcal{EQ}} \subset \mathscr{A}$ and $\mathscr{A}$ is a tractable class, only having Gen-Eq that *do* belong to $\mathscr{A}$ together with $\widehat{\mathcal{EQ}}$ cannot lead to #P-hardness (if #P does not collapse to P). However, we show that having Gen-Eq of arity ≥ 3 that *do not* belong to $\mathscr{A}$ together with $\widehat{\mathcal{EQ}}$ will lead to #P-hardness. There are two scenarios, Corollary 8.14 and Lemma 8.16, that will let us apply Lemma 8.13. Lemma 8.16 says that a nondegenerate Gen-Eq of arity ≥ 3 that does not belong to $\mathscr{A}$ together with $\widehat{\mathcal{EQ}}$ will lead to #P-hardness. Corollary 8.14 says that even if this Gen-Eq of arity ≥ 3 may belong to $\mathscr{A}$, if there is another binary nondegenerate Gen-Eq that does not belong to $\mathscr{A}$, then together with $\widehat{\mathcal{EQ}}$ we still have #P-hardness.

Corollary 8.14 is an immediate corollary of Lemma 8.13, and is also used in Lemma 8.16, so we prove that first.

Corollary 8.14. *Let $a, b, x, y \in \mathbb{C}$. Suppose $f = [x, 0, y]$ and $g = [a, 0, \ldots, 0, b]$ with arity ≥ 3. If $abxy \neq 0$ and $x^4 \neq y^4$, then* Pl-Holant$(\{f, g\} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard.*

Proof. Connect three copies of $f = [x, 0, y]$ to $[1, 0, 1, 0]$, with one on each edge, to get $x[x^2, 0, y^2, 0]$ and we apply Lemma 8.13. □

The proof of Lemma 8.16 applies Corollary 8.14 and an interpolation of a unary signature. We prove this interpolation first; it will be used again later.

Lemma 8.15. *Let $x \in \mathbb{C}$. Suppose $\mathcal{F}$ is a set of signatures containing $f = [1, x, 1]$. If $x \notin \{0, \pm 1\}$ and the matrix form M_f of f has infinite order modulo a scalar, then*

$$\text{Pl-Holant}(\mathcal{F} \cup \{[a, b]\} \cup \widehat{\mathcal{EQ}}) \leq_{\text{T}} \text{Pl-Holant}(\mathcal{F} \cup \widehat{\mathcal{EQ}})$$

for any $a, b \in \mathbb{C}$.

Proof. Consider the unary recursive construction (M_f, s) in Figure 8.1, where $s = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \widehat{\mathcal{EQ}}$. The determinant of $M_f = \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$ is $1 - x^2 \neq 0$. The determinant of $[s\ M_f s] = \begin{bmatrix} 1 & 1 \\ 0 & x \end{bmatrix}$ is $x \neq 0$. By assumption, M_f has infinite order modulo a scalar. Therefore, we can interpolate any unary signature by Lemma 8.7. □

Lemma 8.16. *Let $a, b \in \mathbb{C}$. Suppose $f = [a, 0, \ldots, 0, b]$ with arity ≥ 3. If $ab \neq 0$ and $a^4 \neq b^4$, then* Pl-Holant$(\{f\} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard.*

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Proof. Since $a \neq 0$, we normalize f to $[1, 0, \ldots, 0, x]$, where $x \neq 0$ and $x^4 \neq 1$. If the arity of f is even, then after some number of self-loops, we have $[1, 0, x]$ and are done by Corollary 8.14. Otherwise, the arity of f is odd. After zero or more self-loops, we have $g = [1, 0, 0, x]$. If we have the signature $[1, 1]$, then we can connect this to g to get $[1, 0, x]$ and be done by Corollary 8.14.

We use the signature $[1, x]$, which we obtain via a self-loop on g. Suppose $\Re(x)$, the real part of x, is nonzero. Connecting $[1, x]$ to $[1, 0, 1, 0] \in \widehat{\mathcal{EQ}}$ gives $h = [1, x, 1]$. The eigenvalues of M_h are $\lambda_\pm = 1 \pm x$. They are nonzero because $x \neq \pm 1$. Note that $|x - 1| = |x + 1|$ iff x lies on the pure imaginary line $\Re(x) = 0$, so we have $|\frac{\lambda_+}{\lambda_-}| \neq 1$. Hence the ratio of the eigenvalues is not a root of unity, and so M_h has infinite order modulo a scalar. Therefore, we can interpolate $[1, 1]$ by Lemma 8.15.

Otherwise, $\Re(x) = 0$ but x is not a root of unity since $x \neq \pm \mathrm{i}$. Connecting $[1, x]$ to g gives $h = [1, 0, x^2]$. Clearly $\left(x^2\right)^4 \neq 1$. Hence we apply Corollary 8.14 on h and f, implying that Pl-Holant$(\{f\} \cup \widehat{\mathcal{EQ}})$ is #P-hard. □

8.5 No-Mixing of Tractable Signatures

Perhaps a natural goal at this point would be to attempt a dichotomy Pl-#CSP(f) for a single signature in the form of Pl-Holant$(\{\hat{f}\} \cup \widehat{\mathcal{EQ}})$, which should say that in order *not* to be #P-hard, $\hat{f}$ must belong to $\mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$, the union of the three tractable classes. This will indeed be proved (Theorem 8.32) before we prove the main dichotomy (Theorem 8.34). However, seemingly taking a detour, we will actually prove a no-mixing theorem (Theorem 8.22) first. This no-mixing theorem assumes that a given signature set $\mathcal{F}$ already satisfies $\mathcal{F} \subseteq \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$ (a necessary condition that would be implied by the single signature dichotomy), and states that the problem is still #P-hard unless $\mathcal{F}$ is contained in a single tractable class. In other words, no mixing of distinct tractable classes can occur. The reason we organize the proof in this way is because the no-mixing theorem will be useful in the proof of the single signature dichotomy.

Thus in this section, we prove that various tractable signatures, if they do not belong to a single tractable class, will produce #P-hardness in combination. There are three tractable classes. To help understand these classes, Figure 8.4 contains a Venn diagram of symmetric signatures in $\mathscr{A}$, $\widehat{\mathscr{P}}$, and $\mathscr{M}$. Note that nontrivial intersections occur; but crucially $\mathscr{M} \cap \widehat{\mathscr{P}} \subset \mathscr{A}$, thus there is an empty cell in the Venn diagram

$$\mathscr{M} \cap \widehat{\mathscr{P}} - \mathscr{A} = \emptyset. \tag{8.1}$$

Exercise: Prove (8.1). Thus there is an empty cell in the Venn diagram (8.4).

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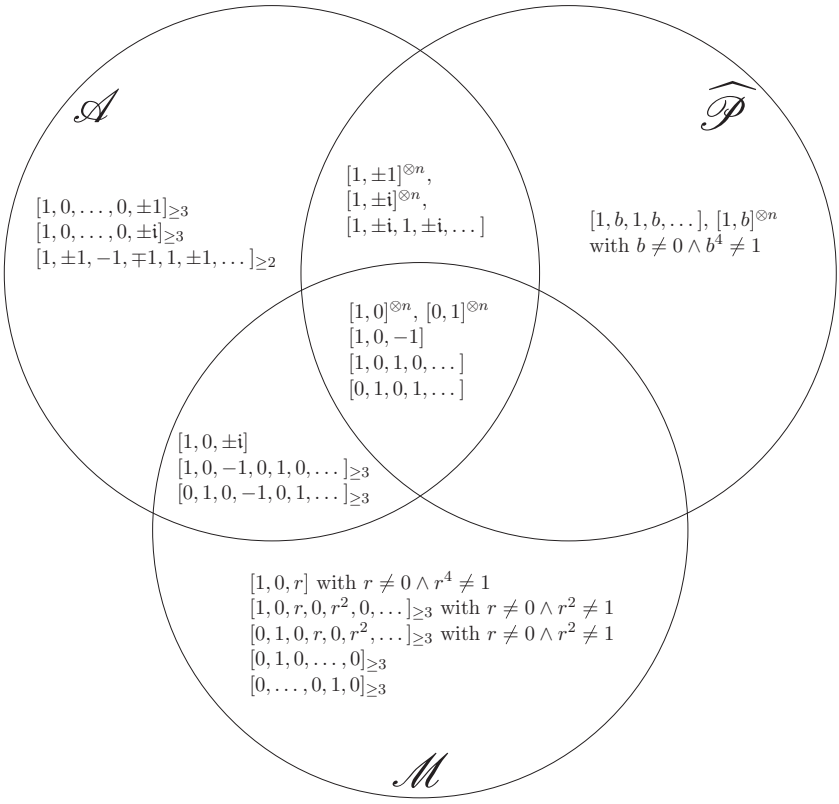

Figure 8.4. Venn diagram of the tractable Pl-#CSP signature sets in the Hadamard basis. Each signature has been normalized for simplicity of presentation. For a signature f, the notation "$f_{\geq k}$" is short for "arity$(f) \geq k$". Notice that $\mathscr{M} \cap \widehat{\mathscr{P}} - \mathscr{A}$ is empty.

Exercise: Verify that all nonzero symmetric signatures in $\mathscr{A}$, $\widehat{\mathscr{P}}$, and $\mathscr{M}$ up to a nonzero scalar are depicted correctly in the Venn diagram (8.4).

We remark that $\mathscr{P}$ (and hence $\widehat{\mathscr{P}}$ also) contains all degenerate signatures, in particular all unary signatures.

We will prove five lemmas giving pairwise no-mixing statements. In the first two lemmas one of the signatures is unary. The last three lemmas consider the general case of the three pairwise no-mixing statements between three tractable classes, respectively.

Lemma 8.17 is a no-mixing lemma of $\mathscr{A}$ versus $\widehat{\mathscr{P}}$, where the unary signature is not in $\mathscr{A}$. Note that for a unary signature $[a, b]$, the condition $ab \neq 0$ and

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$a^4 \neq b^4$ is equivalent to $[a, b] \notin \mathscr{A}$. Also note that since $\mathscr{P}$ contains all unary signatures, so does $\widehat{\mathscr{P}}$.

Lemma 8.17. *Let $a, b \in \mathbb{C}$. Suppose $f \in \mathscr{A} - \widehat{\mathscr{P}}$. If $ab \neq 0$ and $a^4 \neq b^4$, then the problem* Pl-Holant$(\{f, [a, b]\} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard.*

Proof. To understand what forms a symmetric signature $f \in \mathscr{A} - \widehat{\mathscr{P}}$ may take, we may consider a holographic transformation $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Note that f is nondegenerate since $f \notin \widehat{\mathscr{P}}$. As $H_2 \in \text{Stab}(\mathscr{A})$, we have $H_2 f \in \mathscr{A} - \mathscr{P}$. Since $\mathscr{F}_1 \subset \mathscr{P}$, we have $H_2 f \in \mathscr{F}_2 \cup \mathscr{F}_3 - \mathscr{P}$. By analyzing the forms in $\mathscr{F}_2 \cup \mathscr{F}_3 - \mathscr{P}$ we can obtain the following possibilities for f, up to a nonzero scalar.

- $[1, 0, \pm \mathrm{i}]$.
- $[1, 0, \ldots, 0, x]$ of arity ≥ 3 with $x^4 = 1$.
- $[1, \pm 1, -1, \mp 1, 1, \pm 1, -1, \mp 1, \ldots]$ of arity ≥ 2.
- $[1, 0, -1, 0, 1, 0, -1, 0, \ldots, 0 \text{ or } 1 \text{ or } (-1)]$ of arity ≥ 3.
- $[0, 1, 0, -1, 0, 1, 0, -1, \ldots, 0 \text{ or } 1 \text{ or } (-1)]$ of arity ≥ 3.

We handle these cases below. Our general technique is to produce a binary signature that does not fit the tractability criterion of Theorem 8.3 for spin systems. In one case we also use Corollary 8.14 and Lemma 8.16 based on the domain pairing technique.

1. Suppose $f = [1, 0, \pm \mathrm{i}]$. Connecting $[a, b]$ to $[1, 0, 1, 0]$ gives $[a, b, a]$, and connecting two copies of $[1, 0, \pm \mathrm{i}]$ to $[a, b, a]$, one on each edge, gives $g = [a, \pm \mathrm{i}b, -a]$. Then Pl-Holant$(g \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 8.3.
2. Suppose $f = [1, 0, \ldots, 0, x]$ of arity ≥ 3 with $x^4 = 1$. Connecting $[a, b]$ to f gives $g = [a, 0, \ldots, 0, bx]$ of arity ≥ 2. Note that $(bx)^4 = b^4 \neq a^4$. If the arity of g is exactly 2, then Pl-Holant$(\{f, g\} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Corollary 8.14, so we are done. Otherwise, the arity of g is ≥ 3 and Pl-Holant$(\{g\} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 8.16.
3. Suppose $f = [1, \pm 1, -1, \ldots]$ of arity ≥ 2. Connecting some number of $[1, 0]$ gives $[1, \pm 1, -1]$ of arity exactly 2. Connecting $[a, b]$ to $[1, 0, 1, 0]$ gives $[a, b, a]$ and connecting two copies of $[a, b, a]$ to $[1, \pm 1, -1]$, one on each edge, gives $g = [a^2 \pm 2ab - b^2, \pm(a^2 + b^2), -a^2 \pm 2ab + b^2]$. This is easily verified by

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} 1 & \pm 1 \\ \pm 1 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2 \pm 2ab - b^2 & \pm(a^2 + b^2) \\ \pm(a^2 + b^2) & -a^2 \pm 2ab + b^2 \end{bmatrix}.$$

Then Pl-Holant$(g \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 8.3.

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4. Suppose $f = [1, 0, -1, 0, \dots]$ of arity ≥ 3. Connecting some number of $[1, 0]$ gives $g = [1, 0, -1, 0]$ of arity exactly 3. Connecting $[a, b]$ to g gives $h = [a, -b, -a]$. Then Pl-Holant$(h \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 8.3.
5. The argument for $f = [0, 1, 0, -1, \dots]$ is similar to the previous case. □

Before we move to the next lemma, we observe that in Lemma 8.17 we did not explicitly require that $\{f, [a, b]\} \not\subseteq \mathcal{M}$. In general, a pairwise no-mixing statement should say that for some f and g, if $\{f, g\}$ is not contained in a single tractable class, then their joint presence leads to #P-hardness. The condition in Lemma 8.17 ensures that $\{f, [a, b]\} \not\subseteq \mathcal{A}$ and $\{f, [a, b]\} \not\subseteq \widehat{\mathcal{P}}$. However, even though it was not explicitly stated that $\{f, [a, b]\} \not\subseteq \mathcal{M}$, this is implied since $ab \neq 0$ violates the parity requirement for $[a, b] \in \mathcal{M}$.

The next lemma is a no-mixing statement of $\mathcal{M}$ versus $\mathcal{A}$ involving a unary signature $[a, b]$. Note that for a unary signature $[a, b]$, the condition $ab \neq 0$ is equivalent to $[a, b] \notin \mathcal{M}$. We explicitly require that $f \in \mathcal{M} - \mathcal{A}$, but we do not require that $[a, b] \in \mathcal{A}$. This gives us a slightly wider applicability. Also note that we do not explicitly require that $\{f, [a, b]\} \not\subseteq \widehat{\mathcal{P}}$. This is because $f \in \mathcal{M} - \mathcal{A}$ implies that $f \notin \widehat{\mathcal{P}}$ by the observation (8.1).

Lemma 8.18. *Let $a, b \in \mathbb{C}$. If $f \in \mathcal{M} - \mathcal{A}$ and $ab \neq 0$, then* Pl-Holant$(\{f, [a, b]\} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard.*

Proof. By considering the forms in Theorem 4.11 for symmetric matchgates $\mathcal{M}$ and the list of signatures in $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$ in Section 3.2, it is easy to see that, up to a nonzero scalar, the possibilities for $f \in \mathcal{M} - \mathcal{A}$ are as follows:

- $[1, 0, r]$ with $r \neq 0$ and $r^4 \neq 1$.
- $[1, 0, r, 0, r^2, 0, \dots]$ of arity ≥ 3 with $r \neq 0$ and $r^2 \neq 1$.
- $[0, 1, 0, r, 0, r^2, \dots]$ of arity ≥ 3 with $r \neq 0$ and $r^2 \neq 1$.
- $[0, 1, 0, \dots, 0]$ of arity ≥ 3.
- $[0, \dots, 0, 1, 0]$ of arity ≥ 3.

We handle these cases below. Again our technique is to produce a binary signature that does not fit the tractability criterion of Theorem 8.3 for spin systems.

1. Suppose $f = [1, 0, r]$ with $r^4 \neq 1$ and $r \neq 0$. Connecting $[a, b]$ to $[1, 0, 1, 0]$ gives $[a, b, a]$ and connecting two copies of $[1, 0, r]$ to $[a, b, a]$, one on each edge, gives $g = [a, br, ar^2]$. If $a^2 \neq b^2$, then Pl-Holant$(g \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 8.3.

   Otherwise, $a^2 = b^2$ and we first connect $[a, b]$ to $[1, 0, r]$ to get $[a, br]$. Then by the same construction, we have $g' = [a, br^2, ar^2]$ and Pl-Holant$(g' \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 8.3.

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2. Suppose $f = [1, 0, r, 0, \dots]$ of arity ≥ 3 with $r^2 \neq 1$ and $r \neq 0$. Connecting some number of $[1, 0] \in \widehat{\mathcal{EQ}}$ gives $g = [1, 0, r, 0]$ of arity exactly 3. Connecting $[a, b]$ to g gives $h = [a, br, ar]$. If $a^2 \neq b^2 r$, then Pl-Holant$(h \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 8.3.

   Otherwise, $a^2 = b^2 r$ and we first connect $[1, 0]$ and $[a, b]$ to $[1, 0, r, 0]$ to get $[a, br]$. Then by the same construction, we have $g' = [a, br^2, ar]$ and Pl-Holant$(g' \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 8.3.
3. The argument for $f = [0, 1, 0, r, \dots]$ is similar to the previous case.
4. Suppose $f = [0, 1, 0, \dots, 0]$ of arity $k \geq 3$. Connecting $k - 2$ copies of $[a, b]$ to f gives $g = a^{k-3}[(k-2)b, a, 0]$. Then Pl-Holant$(g \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 8.3.
5. The argument for $f = [0, \dots, 0, 1, 0]$ is similar to the previous case. □

Now we consider the general case of two signatures from two different tractable sets. The three tractable classes $\mathscr{A}$, $\widehat{\mathscr{P}}$ and $\mathscr{M}$ give rise to three pairs of pairwise cases to consider, each of which is covered in one of the next three lemmas.

Lemma 8.19 ($(\mathscr{A}, \widehat{\mathscr{P}})$-no-mixing). *If $f \in \mathscr{A} - \widehat{\mathscr{P}}$ and $g \in \widehat{\mathscr{P}} - \mathscr{A}$, then* Pl-Holant$(\{f, g\} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard.*

Proof. The only possibility for g is $[a, b, a, b, \dots]$, or $[a, b]^{\otimes n}$, where $n \geq 1$, $ab \neq 0$ and $a^4 \neq b^4$. Connecting some number of $[1, 0] \in \widehat{\mathcal{EQ}}$ to g gives $[a, b]$ up to a nonzero scalar, and we are done by Lemma 8.17. □

Note that implicitly the validity of Lemma 8.19 hinges on the fact that (8.1) holds, namely that empty cell in the Venn diagram (8.4). Were this not the case, that signature $g \in \widehat{\mathscr{P}} - \mathscr{A}$ could have been in $\mathscr{M}$, and that $f \in \mathscr{A} - \widehat{\mathscr{P}}$ may also belong to $\mathscr{M}$, then Pl-Holant$(\{f, g\} \cup \widehat{\mathcal{EQ}})$ would be tractable in P and Lemma 8.19 would collapse #P and P.

Similarly if (8.1) were not true, then the following Lemma 8.20 would also callapse #P and P, because that $f \in \mathscr{M} - \mathscr{A}$ could have belonged to $\widehat{\mathscr{P}}$, and it would be possible that $\{f, g\} \cup \widehat{\mathcal{EQ}} \subset \widehat{\mathscr{P}}$.

Lemma 8.20 ($(\mathscr{M}, \mathscr{A})$-no-mixing). *If $f \in \mathscr{M} - \mathscr{A}$ and $g \in \mathscr{A} - \mathscr{M}$, then* Pl-Holant$(\{f, g\} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard.*

Proof. If g is degenerate in $\mathscr{A} - \mathscr{M}$, then g does not contain any 0 entries. Suppose g does not contain a 0 entry. Then after connecting some number of $[1, 0]$ to g, we have a unary signature $[a, b]$ with $ab \neq 0$, and are done by Lemma 8.18.

Otherwise, g contains a 0 entry. In particular, g is nondegenerate. Then, by the forms of $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$ and $\mathscr{M}$, it must be a Gen-Eq (see Figure 8.4),

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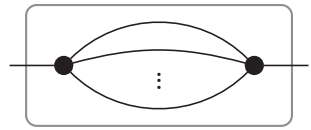

Figure 8.5. The vertices are assigned $f = [0, 1, 0, \ldots, 0]$.

$g = [x, 0, \ldots, 0, y]$ of arity ≥ 3 with $xy \neq 0$ (and $x^4 = y^4$). Up to a nonzero scalar, the possibilities for f are as follows:

- $[1, 0, r]$ with $r \neq 0$ and $r^4 \neq 1$.
- $[1, 0, r, 0, r^2, 0, \ldots]$ of arity ≥ 3 with $r \neq 0$ and $r^2 \neq 1$.
- $[0, 1, 0, r, 0, r^2, \ldots]$ of arity ≥ 3 with $r \neq 0$ and $r^2 \neq 1$.
- $[0, 1, 0, 0, \ldots, 0]$ of arity ≥ 3.
- $[0, \ldots, 0, 0, 1, 0]$ of arity ≥ 3.

We handle these cases below.

1. Suppose $f = [1, 0, r]$ with $r \neq 0$ and $r^4 \neq 1$. Then we are done by Corollary 8.14.
2. Suppose $f = [1, 0, r, 0, \ldots]$ of arity ≥ 3 with $r \neq 0$ and $r^2 \neq 1$. After connecting some number of $[1, 0]$ to f, we have $h = [1, 0, r, 0]$ of arity exactly 3. Then Pl-Holant$(\{g, h\} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 8.13.
3. Suppose $f = [0, 1, 0, r, \ldots]$ of arity ≥ 3 with $r \neq 0$ and $r^2 \neq 1$. After connecting some number of $[1, 0]$ to f, we have $h = [0, 1, 0, r]$ of arity exactly 3. Connecting two more copies of $[1, 0]$ to h gives $[0, 1]$. Then we apply a holographic transformation by $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so g is transformed to $\hat{g} = [y, 0, \ldots, 0, x]$ and h is transformed to $\hat{h} = [r, 0, 1, 0]$. Since T is orthogonal, Pl-Holant$(\{g, h\} \cup \widehat{\mathcal{EQ}})$ is transformed to Pl-Holant$(\{\hat{g}, \hat{h}\} \cup T\widehat{\mathcal{EQ}})$. Every even arity signature in $\widehat{\mathcal{EQ}}$ remains unchanged after a holographic transformation by T. By attaching $[0, 1]T = [1, 0]$ to every even arity signature in $T\widehat{\mathcal{EQ}}$, we obtain all of the odd arity signatures in $\widehat{\mathcal{EQ}}$ again. Then Pl-Holant$(\{\hat{g}, \hat{h}\} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 8.13.
4. Suppose $f = [0, 1, 0, \ldots, 0]$ of arity $k \geq 3$. The gadget in Figure 8.5 with f assigned to both vertices has signature $h = [k - 1, 0, 1]$. Then Pl-Holant$(\{g, h\} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Corollary 8.14.
5. The argument for $f = [0, \ldots, 0, 1, 0]$ is similar to the previous case. □

Note that in Lemma 8.19 and 8.20 we do not explicitly require that $\{f, g\}$ is not contained in the third tractable class. But that is implied (as it should). In Lemma 8.19, $g \in \widehat{\mathscr{P}} - \mathscr{A}$ implies that $g \notin \mathscr{M}$. In Lemma 8.20, $f \in \mathscr{M} - \mathscr{A}$

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implies that $f \notin \widehat{\mathscr{P}}$. Both these statements follow from the observation (8.1). However in Lemma 8.21 we do have to explicitly require that $\{f, g\} \not\subseteq \mathscr{A}$.

Lemma 8.21 ($(\widehat{\mathscr{P}}, \mathscr{M})$-no-mixing). *If $f \in \mathscr{M} - \widehat{\mathscr{P}}$ and $g \in \widehat{\mathscr{P}} - \mathscr{M}$ and $\{f, g\} \not\subseteq \mathscr{A}$, then* Pl-Holant$(\{f, g\} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard.*

Proof. The only possibilities for g are $[a, b, a, b, \dots]$ or $[a, b]^{\otimes n}$ with $ab \neq 0$ (see Figure 8.4). Connecting some number of $[1, 0]$ to g gives $h = [a, b]$ up to a nonzero scalar. If $f \notin \mathscr{A}$, then Pl-Holant$(\{f, h\} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 8.18. Otherwise, $f \in \mathscr{A}$, so $g \notin \mathscr{A}$ and Pl-Holant$(\{f, g\} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 8.19. □

Observe that in the proof of Lemma 8.21 when we apply Lemma 8.18 to $\{f, h\}$, we have $f \in \mathscr{M} - \mathscr{A}$, but we do not necessarily have the condition $h \in \mathscr{A} - \mathscr{M}$. We only have $h \notin \mathscr{M}$. However, Lemma 8.18 does not require that $h \in \mathscr{A}$. Also note that we choose to prove the three lemmas, Lemma 8.19, 8.20, and 8.21, in that order so that the proof of Lemma 8.21 can apply Lemma 8.19.

We summarize this section with the following theorem, which says that the tractable signature sets cannot mix. Signatures from different tractable sets, when put together, lead to #P-hardness.

Theorem 8.22 (No-mixing theorem). *Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. If $\mathcal{F} \subseteq \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$, then* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard unless $\mathcal{F} \subseteq \mathscr{A}$, or $\mathcal{F} \subseteq \widehat{\mathscr{P}}$, or $\mathcal{F} \subseteq \mathscr{M}$, in which case* Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is tractable.*

Proof. If $\mathcal{F}$ is a subset of $\mathscr{A}$, $\widehat{\mathscr{P}}$, or $\mathscr{M}$, then the tractability is given in Theorem 8.4. Suppose $\mathcal{F}$ is not a subset of $\mathscr{A}$, $\widehat{\mathscr{P}}$, or $\mathscr{M}$. Then $\mathcal{F}$ contains a signature $g \in (\widehat{\mathscr{P}} \cup \mathscr{M}) - \mathscr{A}$ since $\mathcal{F} \not\subseteq \mathscr{A}$. As $\mathcal{F} \subseteq \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$, either $\mathcal{F}$ contains a signature $f \in \mathscr{A} - (\widehat{\mathscr{P}} \cup \mathscr{M})$, or $\mathcal{F} \subseteq \widehat{\mathscr{P}} \cup \mathscr{M}$. Suppose it is the first case. If $g \in \widehat{\mathscr{P}} - \mathscr{A}$, then Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 8.19. Otherwise, $g \in \mathscr{M} - \mathscr{A}$ and Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 8.20.

Now suppose it is the second case that $\mathcal{F} \subseteq \widehat{\mathscr{P}} \cup \mathscr{M}$. Since $(\widehat{\mathscr{P}} \cap \mathscr{M}) - \mathscr{A}$ is empty by the observation (8.1) (see Figure 8.4), either $g \in (\widehat{\mathscr{P}} - \mathscr{M}) - \mathscr{A}$ or $g \in (\mathscr{M} - \widehat{\mathscr{P}}) - \mathscr{A}$. Suppose $g \in (\widehat{\mathscr{P}} - \mathscr{M}) - \mathscr{A}$. Since $\mathcal{F} \not\subseteq \widehat{\mathscr{P}}$, there exists a signature $f \in \mathscr{M} - \widehat{\mathscr{P}}$. In this case, Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 8.21. Finally suppose $g \in (\mathscr{M} - \widehat{\mathscr{P}}) - \mathscr{A}$. Since $\mathcal{F} \not\subseteq \mathscr{M}$, there exists a signature $f \in \widehat{\mathscr{P}} - \mathscr{M}$. In this case, Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is also #P-hard again by Lemma 8.21. □

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8.6 Pinning for Planar Graphs

The idea of *pinning* is a common reduction technique between counting problems. For the Boolean #CSP framework, pinning fixes some variables to constants 0 or 1, and is represented by the unary signatures $[1, 0]$ and $[0, 1]$ respectively.

Dyer, Goldberg, and Jerrum [DGJ09] showed that one can pin both to 0 and 1 in the #CSP framework; this is Lemma 3.13. However, this reduction is not planar. As it was explained in Section 8.1, it is not possible to prove a general pinning lemma as Lemma 3.13 for the Pl-#CSP framework, assuming #P does not collapse to P. However, in [CLX10] it is shown that for real-weighted Pl-#CSP this difficultly can be overcome if we first perform a holographic transformation by the Hadamard matrix $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Thus in this Hadamard basis, pinning can be done in a planar way, but this holographic transformation is necessary.

The expression of Pl-#CSP($\mathcal{F}$) in the Hadamard basis is Pl-Holant($\widehat{\mathcal{F}} \cup \widehat{\mathcal{EQ}}$). In $\widehat{\mathcal{EQ}} = \{[1, 0], [1, 0, 1], [1, 0, 1, 0], \ldots\}$ we already have "half" of the pinning pair $[1, 0]$. So pinning in Pl-Holant($\widehat{\mathcal{F}} \cup \widehat{\mathcal{EQ}}$) amounts to obtaining the missing signature $[0, 1]$. We remark that there is no intrinsic precedence of $[1, 0]$ over $[0, 1]$. This is merely an artifact of our choice of $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Recall the equivalence relation $\sim$ defining the basis manifold $\mathcal{B}$, we could have made an equally valid choice $H_2' = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} H_2$. Then $[0, 1] \in H_2'\mathcal{EQ}$ but $[1, 0] \notin H_2'\mathcal{EQ}$. Choosing either H_2 or H_2' gets us one of the two pinning signatures; getting the other "half" of the pair $\{[1, 0], [0, 1]\}$ will be a significant step toward proving the dichotomy theorem Theorem 8.34.

The Road to Pinning

Our goal is to prove a dichotomy for Pl-Holant($\mathcal{C} \cup \widehat{\mathcal{EQ}}$), for $\mathcal{C} = \widehat{\mathcal{F}}$. An important step for the dichotomy of Pl-Holant($\mathcal{C} \cup \widehat{\mathcal{EQ}}$) is to prove that it is equivalent to Pl-Holant^c($\mathcal{C} \cup \widehat{\mathcal{EQ}}$).

We begin the road to pinning with a lemma that assumes the presence of $[0, 0, 1] = [0, 1]^{\otimes 2}$, which is the tensor product of two copies of $[0, 1]$. In our pursuit to realize $[0, 1]$, this may be as close as we can get to directly. For example, suppose every signature in the set has even parity. Recall that a signature has even parity if its support is on entries of even Hamming weight. By a simple parity argument, gadgets constructed with signatures of even parity can only realize signatures of even parity. But obviously $[0, 1]$ does not have even parity.

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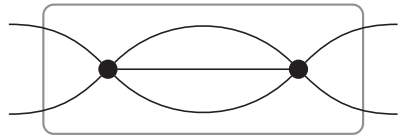

Figure 8.6. The circles are assigned $[a, 0, 0, 0, b, c]$.

However, if every signature has even parity and $[0, 0, 1]$ is present, then we can already prove a dichotomy. To prove this, we first prove the following simple lemma.

Lemma 8.23. *Let $a, b, c \in \mathbb{C}$. If $ab \neq 0$, then* Pl-Holant$([a, 0, 0, 0, b, c])$ *is* #P*-hard.*

Proof. Let f be the signature of the gadget in Figure 8.6 with $[a, 0, 0, 0, b, c]$ assigned to both vertices. The signature matrix of f is

$$\begin{bmatrix} a^2 & 0 & 0 & 0 \\ 0 & b^2 & b^2 & bc \\ 0 & b^2 & b^2 & bc \\ 0 & bc & bc & 3b^2 + c^2 \end{bmatrix},$$

which is redundant. Its compressed form is nonsingular since its determinant is $6a^2b^4 \neq 0$. Thus, we are done by Corollary 7.25. □

Lemma 8.24. *Suppose $\mathcal{F}$ is a set of symmetric signatures with complex weights containing* $[0, 0, 1]$*. If every signature in $\mathcal{F}$ has even parity, then either* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard or $\mathcal{F}$ is a subset of $\mathscr{A}$, $\widehat{\mathscr{P}}$, or $\mathscr{M}$, in which case* Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is tractable.*

Proof. The tractability is given in Theorem 8.4.

Any degenerate symmetric signature having even parity has the form $\lambda[1, 0]^{\otimes n}$, or $\lambda[0, 1]^{\otimes n}$ if the arity n is even. Both belong to $\mathscr{A} \cap \widehat{\mathscr{P}} \cap \mathscr{M}$. In the following we consider nondegenerate signatures of $\mathcal{F}$.

Every signature of arity ≤ 3 satisfying the parity condition is in $\mathscr{M}$. If $\mathcal{F}$ has no signature of arity > 3, then $\mathcal{F} \subseteq \mathscr{M}$ and we are done.

So we may suppose $\mathcal{F}$ contains some nondegenerate signature of arity ≥ 4. For every signature $f \in \mathcal{F}$ with $f = [f_0, f_1, \ldots, f_m]$ and $m \geq 4$, using $[0, 0, 1]$ and $[1, 0]$, we can obtain all subsignatures of the form $[f_{k-2}, 0, f_k, 0, f_{k+2}]$ for any even k such that $2 \leq k \leq m - 2$. If there is any subsignature g of this form that satisfies $f_{k-2}f_{k+2} \neq f_k^2$ and $f_k \neq 0$, then we are done by Corollary 7.25.

Otherwise all subsignatures of the above form satisfy

$$f_{k-2}f_{k+2} = f_k^2 \qquad \text{or} \qquad f_k = 0$$

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for all even k such that $2 \le k \le m-2$. We classify all nondegenerate signatures $f \in \mathcal{F}$ with arity $m \ge 4$ into two types as follows. If there exists an even index $2 \le k \le m-2$ such that $f_k \ne 0$, then by the equation $f_{k-2}f_{k+2} = f_k^2 \ne 0$ we conclude that $f_\ell \ne 0$ for all even indices $0 \le \ell \le m$, and the signature entries of even Hamming weight form a geometric progression. These signatures have the form

$$[\alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n]$$

or (8.2)

$$[\alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n, 0]$$

for some $\alpha, \beta \in \mathbb{C}$, which are in $\mathscr{M}$. Otherwise, $f_k = 0$ for all even indices $2 \le k \le m-2$. Then the signature has the form

$$[x, 0, \ldots, 0, y] \tag{8.3}$$

of even arity $m \ge 4$, where $xy \ne 0$ by being nondegenerate, or

$$[x, 0, \ldots, 0, y, 0] \tag{8.4}$$

of odd arity $m \ge 5$, where $y \ne 0$ by being nondegenerate. We can group the subcase of (8.4) with $x = 0$ as part of the second case of (8.2) with $\alpha = 0$. Then we call signatures in (8.2) type 1 signatures. Type 2 signatures are those in (8.3) and (8.4), with both $x \ne 0$ and $y \ne 0$. Note that there are an odd number of 0's between x and y in type 2 signatures (since they have even parity). If all nondegenerate signatures of arity ≥ 4 in $\mathcal{F}$ are of type 1, then including all signatures of arity ≤ 3 in $\mathcal{F}$, we have $\mathcal{F} \subseteq \mathscr{M}$.

Otherwise $\mathcal{F}$ contains a signature of type 2. Suppose there is some signature $f = [x, 0, \ldots, 0, y, 0]$ of arity ≥ 5 with $xy \ne 0$ in (8.4). After zero or more self-loops, we have $g = [x, 0, 0, 0, y, 0]$ of arity exactly 5. Then we are done by Lemma 8.23.

Otherwise all type 2 signatures in $\mathcal{F}$ are from (8.3). If there is any such $f = [x, 0, \ldots, 0, y]$ of arity ≥ 4 with $xy \ne 0$ *and* $x^4 \ne y^4$, then we are done by Lemma 8.16. Otherwise all type 2 signatures in $\mathcal{F}$ are from (8.3) and satisfy $x^4 = y^4$. This puts every signature in $\mathcal{F}$ of type 2 in $\mathscr{A}$. Therefore $\mathcal{F} \subseteq \mathscr{A} \cup \mathscr{M}$, and by Theorem 8.22, we are done. □

The conclusion of every result in the remainder of Section 8.6 states that we are able to pin (under various assumptions on $\mathcal{F}$). Formally speaking, we repeatedly prove that Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is #P-hard (or in P) if and only if Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is #P-hard (or in P). The difference between these two

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counting problems is the presence of [0, 1] in Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$. We always prove this statement in one of three ways:

1. We show that Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is tractable (so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is as well).
2. Or we show that Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is #P-hard (so Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is as well).
3. Or we show how to reduce Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ to Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ by realizing [0, 1] using signatures in $\mathcal{F} \cup \widehat{\mathcal{EQ}}$.

Note that these three cases together logically imply that

$$\text{Pl-Holant}^c(\mathcal{F} \cup \widehat{\mathcal{EQ}}) \leq_{\mathrm{T}} \text{Pl-Holant}(\mathcal{F} \cup \widehat{\mathcal{EQ}}). \tag{8.5}$$

However, this reduction statement is not proved by directly obtaining the signature [0, 1] with a gadget construction or interpolation in all cases.

Lemma 8.25. *Let $\mathcal{F}$ be any set of complex-weighted symmetric signatures containing* [0, 0, 1]. *Then* Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in P) iff* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in P).*

Proof. If we had a unary signature $[a, b]$ where $b \neq 0$, then connecting $[a, b]$ to [0, 0, 1] gives the signature $[0, b]$, which is [0, 1] after normalizing. Thus, in order to reduce Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ to Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ by constructing [0, 1], it suffices to construct a unary signature $[a, b]$ with $b \neq 0$.

For every signature $f \in \mathcal{F}$ with $f = [f_0, f_1, \ldots, f_m]$, using [0, 0, 1] and [1, 0], we can obtain all subsignatures of the form $[f_{k-1}, f_k]$ for any odd k such that $1 \leq k \leq m$. If any subsignature satisfies $f_k \neq 0$, where k is odd, then we can construct [0, 1].

Otherwise all signatures in $\mathcal{F}$ have even parity and we are done by Lemma 8.24. □

There are two scenarios that lead to Lemma 8.25, which are the focus of the next two lemmas.

Lemma 8.26. *For $x \in \mathbb{C}$, let $\mathcal{F}$ be any set of complex-weighted symmetric signatures containing* $[1, 0, x]$ *such that $x \notin \{0, \pm 1\}$. Then* Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in P) iff* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in P).*

Proof. There are two cases. In either case, we realize [0, 0, 1] and finish by applying Lemma 8.25.

First we claim that the conclusion holds provided the norm $|x| \notin \{0, 1\}$. Combining k copies of $[1, 0, x]$ gives $[1, 0, x^k]$. Since $|x| \notin \{0, 1\}$, x is neither

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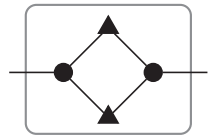

Figure 8.7. The circles are assigned $[1, 0, 1, 0]$ and the triangles are assigned $[1, 0, x]$.

0 nor a root of unity, so we can use polynomial interpolation to realize $[a, 0, b]$ for any $a, b \in \mathbb{C}$, including $[0, 0, 1]$.

Otherwise $|x| = 1$. The gadget in Figure 8.7 has signature $[f_0, f_1, f_2] = [1 + x^2, 0, 2x]$. If $x = \pm\mathrm{i}$, then we have $[0, 0, \pm 2\mathrm{i}]$, which is $[0, 0, 1]$ after normalizing.

Otherwise $x \neq \pm\mathrm{i}$, so $f_0 \neq 0$. Since $x \neq 0$, we have $f_2 \neq 0$. Since $x \neq \pm 1$ and the norm $|x| = 1$, we have $|f_0| < 2$. However, $|f_2| = 2$. Therefore, after normalizing, the signature $[1, 0, y]$ with $y = \frac{2x}{1+x^2}$ has $|y| > 1$, so it can interpolate $[0, 0, 1]$ by our initial claim since $|y| \notin \{0, 1\}$. □

Lemma 8.27. *Let $\mathcal{F}$ be any set of complex-weighted symmetric signatures containing a signature $[f_0, f_1, \ldots, f_n]$ that is not identically zero but has $f_0 = 0$. Then* Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in P) iff* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in P).*

Proof. If $f_1 \neq 0$, then we connect $n - 1$ copies of $[1, 0]$ to f to get $[0, f_1]$, which is $[0, 1]$ after normalizing. If $f_1 = 0$, then $n \geq 2$. If $f_2 \neq 0$, then we connect $n - 2$ copies of $[1, 0]$ to f to get $[0, 0, f_2]$, which is $[0, 0, 1]$ after normalizing. Then we are done by Lemma 8.25. If $f_1 = f_2 = 0$, then $n \geq 3$, and $f = [0, 0, 0, f_3, \ldots, f_n]$. After one or more self-loops, we get a signature with exactly one or two initial 0's, which is one of the above scenarios. □

As a significant step toward pinning for any signature set $\mathcal{F}$, we show how to pin given any binary signature. Some cases resist pinning and are excluded.

Lemma 8.28. *Let $\mathcal{F}$ be any set of complex-weighted symmetric signatures containing a binary signature f. Then* Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in P) iff* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in P) unless $f \in \{[0, 0, 0], [1, 0, -1], [1, r, r^2], [1, b, 1]\}$, up to a nonzero scalar, for any $b, r \in \mathbb{C}$.*

Proof. Let $f = [f_0, f_1, f_2] \neq [0, 0, 0]$. If $f_0 = 0$ then we are done by Lemma 8.27. Otherwise $f_0 \neq 0$, and we normalize f_0 to 1. If Pl-Holant$(f \mid \widehat{\mathcal{EQ}})$

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is #P-hard by Theorem 8.3, then Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) is also #P-hard. Otherwise, f is one of the tractable cases, which implies that

$$f \in \{[1, r, r^2], [1, 0, x], [1, \pm 1, -1], [1, b, 1]\}.$$

If $f = [1, \pm 1, -1]$, then we connect f to $[1, 0, 1, 0] \in \widehat{\mathcal{EQ}}$ to get $[0, \pm 2]$, which is $[0, 1]$ after normalizing. If $f = [1, 0, x]$, then we are done by Lemma 8.26 unless $x \in \{0, \pm 1\}$. The remaining cases are all excluded by assumption, so we are done. □

Pinning in the Hadamard Basis

Before we show how to pin in the Hadamard basis, we handle two simple cases.

Lemma 8.29. *If $\mathcal{F}$ is a set of signatures containing* $[1, \mathrm{i}]$ *or* $[1, -\mathrm{i}]$*, then we have*

$$\text{Pl-Holant}^c(\mathcal{F} \cup \widehat{\mathcal{EQ}}) \leq_{\mathrm{T}} \text{Pl-Holant}(\mathcal{F} \cup \widehat{\mathcal{EQ}}).$$

Proof. Connect two copies of $[1, \pm \mathrm{i}]$ to $[1, 0, 1, 0] \in \widehat{\mathcal{EQ}}$ to get $[0, \pm 2\mathrm{i}]$, which is $[0, 1]$ after normalizing. □

The next lemma considers the signature $[1, b, 1, b^{-1}]$, which we will also use in Theorem 8.32, the single signature dichotomy.

Lemma 8.30. *Let $b \in \mathbb{C}$. If $b \notin \{0, \pm 1\}$, then* Pl-Holant($\{[1, b, 1, b^{-1}]\} \cup \widehat{\mathcal{EQ}}$) *is* #P*-hard.*

Proof. Connect two copies of $[1, 0]$ to $f = [1, b, 1, b^{-1}]$ to get $[1, b]$. Connecting this back to f gives $g = [1 + b^2, 2b, 2]$. Then Pl-Holant($g \mid \widehat{\mathcal{EQ}}$) is #P-hard by Theorem 8.3. □

Now we are ready to prove our pinning result.

Theorem 8.31 (Pinning)**.** *Let $\mathcal{F}$ be any set of complex-weighted symmetric signatures. Then* Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard (or in P) iff* Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) *is* #P*-hard (or in P).*

This theorem does not exclude the possibility that either framework contains a problem of intermediate complexity, i.e., a problem that is neither in P nor #P-hard. It only says that if one framework does not contain a problem of intermediate complexity, then the other framework also does not. We will indeed prove neither happens. Our goal is to prove a dichotomy for Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$). By Theorem 8.31, this is equivalent to proving a dichotomy for Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$.

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Proof of Theorem 8.31. For simplicity, we normalize the first nonzero entry of every nonzero signature in $\mathcal{F}$ to 1.

If $\mathcal{F}$ does not contain any nondegenerate signature, then $\mathcal{F} \subseteq \widehat{\mathscr{P}}$ and we are done. Otherwise, let $f \in \mathcal{F}$ be any nondegenerate signature. Being nondegenerate it has arity ≥ 2. We connect some number of $[1, 0]$ to f until we obtain a signature with arity exactly two. We call the resulting signature the binary prefix of f. If this binary prefix is not one of the exceptional forms in Lemma 8.28, then we are done, so assume that it is one of the exceptional forms.

Now we perform case analysis according to the exceptional forms in Lemma 8.28. There are five cases below because we consider $[1, r, r^2]$ as $[1, 0, 0]$ and $[1, r, r^2]$ with $r \neq 0$ as separate cases. In each case, we either show that the conclusion of the theorem holds or that $f \in \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$. After the case analysis, we then handle all of these tractable f together.

1. Suppose the binary prefix of f is $[0, 0, 0]$. Since f is not identically 0, and $f_0 = 0$, we are done by Lemma 8.27.
2. Suppose the binary prefix of f is $[1, 0, -1]$. If f is not of the form

$$[1, 0, -1, 0, 1, 0, -1, 0, \ldots, 0 \text{ or } 1 \text{ or } (-1)], \tag{8.6}$$

   then in particular f has arity ≥ 3 and after one self-loop, we have a signature of arity ≥ 1 with 0 as its first entry but is not identically 0, so by Lemma 8.27 we are done.

   Thus, in this case, we may assume f has the form given in (8.6).
3. Suppose the binary prefix of f is $[1, 0, 0]$. Since f is assumed to be nondegenerate, $f \neq [1, 0, \ldots, 0]$, then after connecting $[1, 0]$ for some $k \geq 0$ times, we have $[1, 0, \ldots, 0, x]$ of arity ≥ 3, where $x \neq 0$. If $x^4 \neq 1$, then Pl-Holant$(\{f\} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 8.16, so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.

   Otherwise, $x^4 = 1$. Suppose that x is not the last entry in f. Then $k \geq 1$. By connecting one fewer $[1, 0]$ than before, we have $g = [1, 0, \ldots, 0, x, y]$ and there are two cases to consider. If the index of x in g is odd, then after zero or more self-loops, we have $h = [1, 0, 0, x, y]$. The determinant of the compressed signature matrix of h is $-2x^2 \neq 0$. Thus, Pl-Holant(h) is #P-hard by Corollary 7.25, so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.

   Otherwise, the index of x in g is even. After zero or more self-loops, we have $h = [1, 0, 0, 0, x, y]$. Then by Lemma 8.23, Pl-Holant(h) is #P-hard, so Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.

   Thus, in this case, we may assume $f = [1, 0, \ldots, 0, x]$ with $x^4 = 1$.
4. Suppose the binary prefix of f is $[1, r, r^2]$, where $r \neq 0$. Since f is nondegenerate, $f \neq [1, r, \ldots, r^n]$, then after connecting some number of $[1, 0]$,

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we have $[1, r, \ldots, r^m, y]$, where $y \neq r^{m+1}$ and $m \geq 2$. Using $[1, 0]$, we can get $[1, r]$. If $r = \pm\mathrm{i}$, then we are done by Lemma 8.29, so assume that $r \neq \pm\mathrm{i}$. Then we can attach $[1, r]$ back to the signature $[1, r, \ldots, r^m, y]$ some number of times to get $g = [1, r, r^2, x]$ after factoring out the normalizing constant $1 + r^2 \neq 0$, where $x \neq r^3$. We connect $[1, r]$ once more to get $h = [1 + r^2, r(1 + r^2), r^2 + rx]$. If h does not have one of the exceptional forms in Lemma 8.28, then we are done, so assume that it does.

Since the second entry of h is not 0 and $x \neq r^3$, the only possibility is that h has the form $[1, b, 1]$ up to a scalar. This gives $x = r^{-1}$. Note that $r \neq \pm 1$ since $x \neq r^3$. A self-loop on $g = [1, r, r^2, r^{-1}]$ gives $[1 + r^2, r + r^{-1}]$, which is $[1, r^{-1}]$ after normalizing. Connecting this back to g gives $h' = [2, 2r, r^2 + r^{-2}]$. We assume that h' has one of the exceptional forms in Lemma 8.28 since we are done otherwise. Since the middle entry $2r \neq 0$, the only possibilities for $h'/2 = [1, r, (r^2 + r^{-2})/2]$ are $[1, r, r^2]$ or $[1, b, 1]$. If it has the form $[1, r, r^2]$, then $(r^2 + r^{-2})/2 = r^2$ thus $r^4 = 1$, a contradiction. So it must be of the form $[1, b, 1]$. But then $(r^2 + r^{-2})/2 = 1$ thus $r^2 = 1$, which is also a contradiction.

5. Suppose the binary prefix of f is $[1, b, 1]$. If $b = \pm 1$, then this binary prefix is degenerate. This case has already been considered in the previous item. So we assume $b \neq \pm 1$. If f is not of the form $[1, b, 1, b, \ldots]$, then first suppose that the index of the first entry in f to break the pattern is even. Then after connecting some number of $[1, 0]$, we have $[1, b, 1, \ldots, b, y]$ of arity ≥ 4, where $y \neq 1$. Then after some number of self-loops and normalizing, we have $g = [1, b, 1, b, x]$, where $x \neq 1$. The determinant of its compressed signature matrix is $2(b^2 - 1)(1 - x) \neq 0$. Thus, Pl-Holant(g) is #P-hard by Corollary 7.25, so Pl-Holant$(\mathcal{F} \cup \widetilde{\mathcal{EQ}})$ is also #P-hard.

Otherwise, the index of the first entry in f to break the pattern is odd. Then after connecting some number of $[1, 0]$, we have $[1, b, 1, \ldots, 1, y]$ of arity ≥ 3, where $y \neq b$. Then after some number of self-loops and normalizing, we have $[1, b, 1, x]$, where $x \neq b$. We do a self-loop to get $g = [2, b + x]$. If $b = 0$, then connecting g to $[1, 0, 1, x]$ gives $h = [2, x, 2 + x^2]$. We assume that h has one of the exceptional forms in Lemma 8.28 since we are done otherwise. Because $x \neq 0$, the only possibility is that h has the form $[1, r, r^2]$ up to a scalar. Then we get $x^2 = -4$, so $g = [2, x] = 2[1, \pm\mathrm{i}]$ and we are done by Lemma 8.29. We use the signature g again below.

Otherwise, $b \neq 0$. Using $[1, 0]$, we can get $h = [1, b, 1]$. As $b \neq \pm 1$, h is nondegenerate. If M_h has infinite order modulo a scalar, then we can interpolate $[0, 1]$ by Lemma 8.15 since $b \notin \{0, \pm 1\}$. If the signature matrix M_h of h has finite order modulo a scalar, then $M_h^\ell = \beta I_2$ for some positive integer ℓ and some nonzero complex value β. Thus after normalizing, we

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can construct the anti-gadget $[1, -b, 1]$ by connecting $\ell - 1$ copies of h together. Connecting $[1, 0]$ to $[1, -b, 1]$ gives $[1, -b]$ and connecting this to $[1, b, 1, x]$ gives $[1 - b^2, 0, 1 - bx]$. If $\frac{1-bx}{1-b^2} \notin \{0, \pm 1\}$, then by Lemma 8.26 we are done.

Otherwise, $y = \frac{1-bx}{1-b^2} \in \{0, \pm 1\}$. For $y = 0$, we get $x = b^{-1}$ and by Lemma 8.30 we are done since we have $[1, b, 1, b^{-1}]$ and $b \notin \{0, \pm 1\}$. For $y = 1$, we get $b = x$, a contradiction. For $y = -1$, we get $2 - b^2 - bx = 0$. Then connecting $g = [2, b + x]$ to $[1, -b, 1]$ gives $[2 - b^2 - bx, x - b] = [0, x - b]$, which is $[0, 1]$ after normalizing since $x \neq b$.

Thus, in this case, we may assume $f = [1, b, 1, b, \dots]$.

At this point, every nondegenerate signature in $\mathcal{F}$ must be of one of the following forms:

- $[1, 0, -1, 0, 1, 0, -1, 0, \dots, 0 \text{ or } 1 \text{ or } (-1)]$, which is in $\mathcal{A} \cap \mathcal{M}$.
- $[1, 0, \dots, 0, x]$, where $x^4 = 1$, which is in $\mathcal{A}$.
- $[1, b, 1, b, \dots, 1 \text{ or } b]$, which is in $\widehat{\mathcal{P}}$.

Also every degenerate signature is in $\widehat{\mathcal{P}}$. Therefore $\mathcal{F} \subseteq \mathcal{A} \cup \widehat{\mathcal{P}} \cup \mathcal{M}$ and we are done by Theorem 8.22, the no-mixing theorem. □

8.7 Planar #CSP Dichotomy

In this section, we prove our main dichotomy theorem of this chapter. We begin with a dichotomy for a single signature.

Recall that Pl-Holant$(\{\hat{f}\} \cup \widehat{\mathcal{EQ}})$ is the equivalent expression of Pl-#CSP$(\{f\})$, where $\hat{f} = H_2 f$. Theorem 8.32 is expressed in the Pl-Holant language.

Theorem 8.32. *If f is a nondegenerate symmetric signature of arity ≥ 2 with complex weights in Boolean variables, then* Pl-Holant$(\{f\} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard unless $f \in \mathcal{A} \cup \widehat{\mathcal{P}} \cup \mathcal{M}$, in which case the problem is computable in polynomial time.*

Proof. When $f \in \mathcal{A} \cup \widehat{\mathcal{P}} \cup \mathcal{M}$, the problem is tractable by Theorem 8.4. When $f \notin \mathcal{A} \cup \widehat{\mathcal{P}} \cup \mathcal{M}$, we prove that Pl-Holant$^c(\{f\} \cup \widehat{\mathcal{EQ}})$ is #P-hard, which is equivalent because of pinning (Theorem 8.31). The advantage of Pl-Holant^c is that we can use $[1, 0]$ and $[0, 1]$ to obtain any subsignature of f.

Notice that once we have $[0, 1]$ and $\widehat{\mathcal{EQ}}$, we can realize every signature in $T\widehat{\mathcal{EQ}}$, where $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. In fact, every even arity signature in $\widehat{\mathcal{EQ}}$ is also in

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$T\widehat{\mathcal{EQ}}$, and we obtain all the odd arity signatures in $T\widehat{\mathcal{EQ}}$ by attaching $[0, 1]$ to all the even arity signatures in $\widehat{\mathcal{EQ}}$. Also note that $T[0, 1] = [1, 0]$ and $T[1, 0] = [0, 1]$. Therefore, Pl-Holant$^c(\{f\} \cup \widehat{\mathcal{EQ}}) \equiv_{\mathrm{T}}$ Pl-Holant$^c(\{Tf\} \cup \widehat{\mathcal{EQ}})$. Furthermore, $\mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$ is closed under T. We use these facts later.

The possibilities for f can be divided into three cases:

- f satisfies the parity condition.
- f does not satisfy the parity condition but does contain a 0 entry.
- f does not contain a 0 entry.

We handle these cases below.

1. Suppose that f satisfies the parity condition. If f has even parity, then by Lemma 8.24 we are done.

   Otherwise, f has odd parity. If f has odd arity, $f = [0, f_1, 0, \ldots, 0, f_{2k+1}]$, then under a holographic transformation by $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, f is transformed to $f' = [f_{2k+1}, 0, \ldots, 0, f_1, 0]$, which has even parity. Then either Pl-Holant$^c(\{f'\} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 8.24 (and thus Pl-Holant$^c(\{f\} \cup \widehat{\mathcal{EQ}})$ is also #P-hard), or $f' \in \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$ (and thus $f \in \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$).

   Otherwise, f has odd parity and even arity, $f = [0, f_1, 0, \ldots, 0, f_{2k-1}, 0]$. Connect $[0, 1]$ to f to get a signature $g = [f_1, 0, \ldots, 0, f_{2k-1}, 0]$ with even parity and odd arity. Then either Pl-Holant$^c(\{g\} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 8.24 (and thus Pl-Holant$^c(\{f\} \cup \widehat{\mathcal{EQ}})$ is also #P-hard), or $g \in \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$. In the latter case, the last entry of g is 0, since g has odd arity and even parity. Then inspection of Figure 8.4 shows that $g \in \mathscr{M}$. In particular, the even parity entries of g form a geometric progression, by Theorem 4.11 (case 2, this includes the form $[0, \ldots, 0, *, 0]$). Therefore $f \in \mathscr{M}$ since f has odd parity and the same geometric progression among its odd parity entries.
2. Suppose that f does not satisfy the parity condition but does contain a 0 entry. Then there are two nonzero entries f_i and f_j, where $i < j$ and i and j are of the opposite parity. Take two such entries with minimum $j - i$, which is odd. If $j - i > 1$ then any entry f_ℓ with $i < \ell < j$ must be 0, since $j - i = (j - \ell) + (\ell - i)$ and one of them must be odd. Thus f_i and f_j are separated by an even number $j - i - 1 \geq 0$ of 0 entries. And so f contains a subsignature $g = [a, 0, \ldots, 0, b]$ of odd arity $n = 2k + 1 \geq 1$, where $ab \neq 0$.

   If $k = 0$, then $n = 1$ and $g = [a, b]$. But since f does contain a 0 entry, f has arity greater than one. Then we can shift from the positions of g either

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to the right or to the left and find a 0 entry in f and obtain (by pinning) a binary subsignature h of the form $[c, d, 0]$ or $[0, c, d]$, where $cd \neq 0$. Then Pl-Holant($h \mid \widehat{\mathcal{EQ}}$) is #P-hard by Theorem 8.3, so Pl-Holant($\{f\} \cup \widehat{\mathcal{EQ}}$) is also #P-hard, by applying Theorem 8.31.

Otherwise $k \geq 1$, so $n \geq 3$. If $a^4 \neq b^4$, then Pl-Holant($\{g\} \cup \widehat{\mathcal{EQ}}$) is #P-hard by Lemma 8.16, so Pl-Holant($\{f\} \cup \widehat{\mathcal{EQ}}$) is also #P-hard, by applying Theorem 8.31.

Otherwise, $a^4 = b^4$, so $g \in \mathscr{A}$. If $f = g$, then we are done, so assume that $f \neq g$, which implies that there is another entry just before a or just after b. If this entry is nonzero, then f has a subsignature h of the form $[c, a, 0]$ ($ac \neq 0$) or $[0, b, d]$ ($bd \neq 0$). Then Pl-Holant($h \mid \widehat{\mathcal{EQ}}$) is #P-hard by Theorem 8.3, so Pl-Holant($\{f\} \cup \widehat{\mathcal{EQ}}$) is also #P-hard.

Otherwise, this entry is 0 and f has a subsignature h of the form

$$[0, a, 0, \ldots, 0, b] \qquad \text{or} \qquad [a, 0, \ldots, 0, b, 0]$$

of arity ≥ 4, whereas $g = [a, 0, \ldots, 0, b]$ is a subsignature of h. Since g has odd arity, the arity of h is even. Then after some number of self-loops, we have a signature h' of the form $[0, a, 0, 0, b]$ or $[a, 0, 0, b, 0]$ of arity exactly 4. Then Pl-Holant(h') is #P-hard by Corollary 7.25 since $ab \neq 0$, so Pl-Holant($\{f\} \cup \widehat{\mathcal{EQ}}$) is also #P-hard.

3. Suppose f contains no 0 entry. If f has a binary subsignature g such that Pl-Holant($g \mid \widehat{\mathcal{EQ}}$) is #P-hard by Theorem 8.3, then Pl-Holant($\{f\} \cup \widehat{\mathcal{EQ}}$) is also #P-hard.

   Otherwise every binary subsignature $[a, b, c]$ of f satisfies the conditions of some tractable case in Theorem 8.3. Since there are no 0 entries, there are three possible tractable cases for $[a, b, c]$:

   (D) it is degenerate with condition $ac = b^2$ (case 1);
   (A) it has the affine type $\mathscr{A}$ with condition $[ac = -b^2] \wedge [a = -c]$ (case 3); or
   (P) it has a Hadamard-transformed product type $\widehat{\mathscr{P}}$ with condition $a = c$ (case 4).

   Remark: Condition (D) refers to degenerate. Condition (A) refers to a subset of affine signatures. Some affine signatures are included in condition (P) for $\widehat{\mathscr{P}}$, e.g., $[1, \mathrm{i}, 1]$. To group them this way is for the technical convenience in the following proof.

   If every binary subsignature $[a, b, c]$ of f satisfies $ac = b^2$, then f is degenerate, a contradiction. If every binary subsignature $[a, b, c]$ of f satisfies $[ac = -b^2] \wedge [a = -c]$, then $f = [1, \pm 1, -1, \mp 1, 1, \pm 1, -1,$

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$\mp 1, \dots] \in \mathscr{A}$ (up to a nonzero scalar) and we are done. If every binary subsignature $[a, b, c]$ of f satisfies $a = c$, then $f \in \widehat{\mathscr{P}}$ and we are done.

Otherwise, no single one of the three tractable conditions (D), (A) or (P) is satisfied by all binary subsignatures of f. Hence f has arity $n \geq 3$. Let $h_i = [f_i, f_{i+1}, f_{i+2}]$ for all $0 \leq i \leq n-2$ be binary subsignatures of f. Suppose there exists an i such that h_i satisfies the condition (A) (case 3). We claim that there must exist two successive subsignatures $h = [a, b, c]$ and $h' = [b, c, d]$ such that h satisfies condition (A) and $h' = [b, c, d]$ satisfies either condition (D) or condition (P), up to a transformation of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This is because we can start from h_i and search in both directions h_{i-1} and h_{i+1} until we found such h and h'. It is always successful because not all h_j satisfies condition (A). Let $g = [a, b, c, d]$ be the ternary subsignature of f. Then for either case (D) or (P) of h', we have $g = [1, \varepsilon, -1, \varepsilon]$ after normalization, where $\varepsilon^2 = 1$. Connecting two copies of $[0, 1]$ to g gives $[-1, \varepsilon]$. Connecting this back to g gives $g' = [0, -2\varepsilon, 2]$. Then Pl-Holant$(g' \mid \widehat{\mathcal{EQ}})$ is #P-hard by Theorem 8.3, so Pl-Holant$(\{f\} \cup \widehat{\mathcal{EQ}})$ is also #P-hard.

Now we suppose no h_i satisfies condition (A). Then every h_i satisfies either (D) or (P), but no single condition (D) or (P) is satisfied by all. Therefore some h_i must not satisfy (D), which must satisfy (P), and some other h_j must satisfy (D); otherwise all would satisfy (P). Starting from h_i we can shift either to the right or to the left to find the first h_j satisfying (D). Up to a transformation $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we may assume $j > i$. Then h_{j-1} does not satisfy (D), therefore h_{j-1} must satisfy (P). Thus there exists a ternary subsignature $g = [a, b, c, d]$ of f such that $h = [a, b, c]$ satisfies (P) but not (D), and $h' = [b, c, d]$ satisfies (D). Then $g = [1, b, 1, b^{-1}]$ after normalization, where $b^2 \neq 1$. Then Pl-Holant$(\{g\} \cup \widehat{\mathcal{EQ}})$ is #P-hard by Lemma 8.30, so Pl-Holant$(\{f\} \cup \widehat{\mathcal{EQ}})$ is also #P-hard. □

Now we are ready to prove our main dichotomy theorem of this chapter.

Theorem 8.33. *Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then* Pl-Holant$(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ *is* #P*-hard unless $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \widehat{\mathscr{P}}$, or $\mathcal{F} \subseteq \mathscr{M}$, in which case the problem is computable in polynomial time.*

Proof. The tractability is given in Theorem 8.4. When $\mathcal{F}$ is not a subset of $\mathscr{A}$, $\widehat{\mathscr{P}}$, or $\mathscr{M}$, we prove that Pl-Holant$^c(\mathcal{F} \cup \widehat{\mathcal{EQ}})$ is #P-hard, which is sufficient because of pinning (Theorem 8.31).

If $\mathcal{F} \not\subseteq \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$, since $\widehat{\mathscr{P}}$ contains all degenerate signatures (in particular all unary signatures), there exists some nondegenerate signature $f \in \mathcal{F} - (\mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M})$ of arity ≥ 2. Then the problem is #P-hard by

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Theorem 8.32. Otherwise, $\mathcal{F} \subseteq \mathcal{A} \cup \widehat{\mathcal{P}} \cup \mathcal{M}$ and we are done by Theorem 8.22. □

Since Pl-#CSP($\mathcal{F}$) $\equiv_T$ Pl-Holant($\widehat{\mathcal{F}} \cup \widehat{\mathcal{EQ}}$), where $\widehat{\mathcal{F}} = H_2\mathcal{F}$, and $\widehat{\mathcal{EQ}} = H_2\mathcal{EQ}$, we can translate Theorem 8.33 to obtain the main theorem of this chapter, Theorem 8.34, the Pl-#CSP dichotomy. Recall that $H_2\mathcal{A} = \mathcal{A}$, $H_2\mathcal{P} = \widehat{\mathcal{P}}$ and $H_2\mathcal{M} = \widehat{\mathcal{M}}$. Theorem 8.34 is Theorem 8.33 stated in the standard basis.

Theorem 8.34. *Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then* Pl-#CSP($\mathcal{F}$) *is* #P*-hard unless $\mathcal{F} \subseteq \mathcal{A}$, $\mathcal{F} \subseteq \mathcal{P}$, or $\mathcal{F} \subseteq \widehat{\mathcal{M}}$, in which case the problem is computable in polynomial time.*

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