Exercise 1. Let G be a graph of order n ($n \geq 4$), let $\Delta = \Delta(G)$, and let n_i denote the number of vertices of degree i in G. Prove that

(a) if G is a triangulation, then

$$3n_3 + 2n_4 + n_5 = n_7 + 2n_8 + \cdots + (\Delta - 6)n_\Delta + 12;$$

(b) if G is a tree, then

$$n_1 = n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_\Delta + 2.$$

Pf: Let $n = v(G)$, $m = e(G)$. $\phi = \phi(G)$.

(a) Since G is a triangulation,

$$\sum_{f \in F} d_G(f) = 3\phi = 2m$$

By Euler's formula,

$$n + \phi - m = 2$$

Thus,

$$m = 3n - 6$$

By handshaking lemma,

$$\sum_{v \in V(G)} d_G(v) = 2m = 6n - 12$$

We assert that $\delta \geq 3$, because

① there's no vertices of degree 0, otherwise G is unconnected.

② there's no vertices of degree 1, otherwise there's a cut edge which can't appear in triangulation.

③ there's no vertices of degree 2, otherwise there exists such v of degree 2. Let $N_G(v) = \{a, b\}$. Since faces on either side of v are triangle, they must be vab. And there are no vertices in

or out of vab. Thus $n = 3 < 4$, A CONTRADICTION.

Above all,

$$3n_3 + 2n_4 + n_5 = n_7 + 2n_8 + \cdots + (\Delta - 6)n_\Delta + 12.$$

(b) By Euler's formula,

$$n + \phi - m = 2.$$

Since G is a tree,

$$\phi = 1.$$

Thus $m = n - 1$.

By handshaking Lemma,

$$\sum_{v \in V(G)} d_G(v) = 2m = 2n - 2$$

$$n_1 = n_3 + 2n_4 + \cdots + (\Delta - 2)n_\Delta + 2. \qquad \square$$

Exercise 2. Prove that

(a) if G is a connected planar graph of order n and girth $g \geq 3$, then

$$e(G) \leq \frac{g(n-2)}{g-2};$$

(b) the Petersen graph is non-planar.

(a) Since each face's bound is a closed walk containing one circle,

$$g \cdot \phi(G) \leq \sum_{f \in F(G)} d_G(f) = 2e(G).$$

By Euler's formula,

$$n + \phi(G) - e(G) = 2.$$

Hence,

$g \cdot (e(G) + 2 - n) \leq 2 \cdot e(G)$

$$e(G) \leq \frac{g(n-2)}{g-2}$$

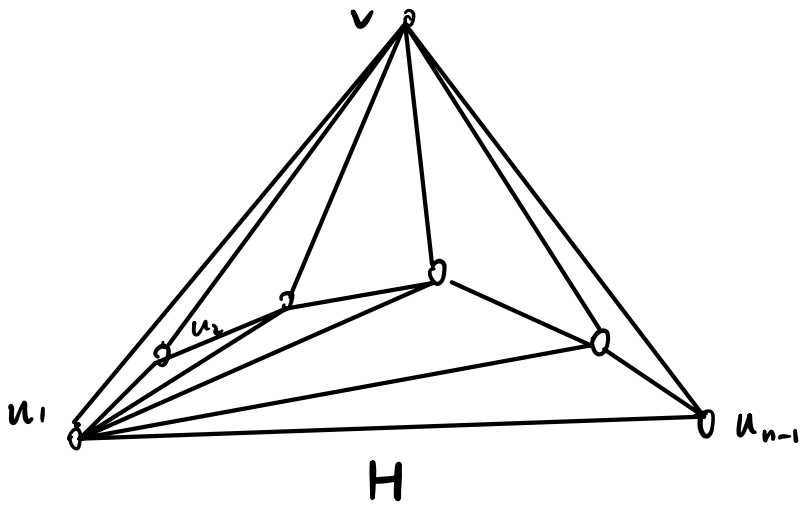
(b) Petersen graph's girth is 5.

Thus $e(G) = 15 > \frac{g(n-2)}{g-2} = \frac{5 \times 8}{3} = \frac{40}{3}$.

Hence Peterson graph is non-planar. □

Exercise 3. Let G be a simple planar graph with $\Delta(G) = n - 1$. Prove that if $n \geq 5$, then G contains two nonadjacent vertices both of degree at most 3.

Pf:



We consider constructing a maximal planar graph H containing G.

Since $\Delta(G) = n-1$, choose v with $d_G(v) = n-1$.

Let $N_G(v) = \{u_1, u_2, \cdots u_{n-1}\}$.

Since H is maximal, $u_1u_2, u_2u_3, \cdots, u_{n-2}u_{n-1}, u_{n-1}u_1$ are in $E(H)$.

And $H - v$ is also a triangulation.

We need to find 2 nonadjacent vertices of degree 2 in $H-V$, then G contains 2 nonadjacent vertices of degree at most 3.

Since there are $n-3$ inner triangles in $H-V$, $n-4$ inner edges in $H-V$, the dual graph of $n-3$ inner triangles is a tree.

Hence there's at least 2 leaf vertices in it, which refer to 2 triangles with exact one inner edge in $H-V$. The incident vertex of two boundary edge is of degree 2 in $H-V$.

By now, we find two vertices of degree 2 in $H-V$, which are two vertices of degree at most 3 in G.

If they are adjacent, then $H-V$ must be a triangle, $n-1=3 \Rightarrow n=4<5$,

A CONTRADICTION.

In conclusion, the original proposition is correct. □

Exercise 4. Prove that

(a) if G is non-planar, then every subdivision of G is non-planar;

(b) if G is planar, then G contains no subdivision of K_5 or $K_{3,3}$;

(c) if G is planar, then every subgraph of G is planar.

pf:

(a)

Let H be a subdivision of G, and suppose that H is planar. Take a plane drawing $\tilde{H}$ of H. Replace each path corresponding to an edge of G by a single edge drawn along the same curve. This gives a plane drawing of G. Thus G is planar, A CONTRADICTION. Hence if G is nonplanar, then every subdivision of G is nonplanar.

(b)

If G contains a subdivision of K_5 or $K_{3,3}$, then G contains a nonplanar graph, thus G is nonplanar, A CONTRADICTION. Hence G contains no subdivision of K_5 or $K_{3,3}$.

(c)

Assume G is planar. Let $\tilde{G}$ be the plane graph of G. Let H be a subgraph of G. Delete all vertices and edges of $\tilde{G}$ that do not belong to H, the remaining drawing is a plane representation of H. Hence H is planar. □

Exercise 5. Let G be a plane graph of odd order. Prove that if G has a Hamiltonian cycle, then G has an even number (≥ 2) of odd-degree faces.

Pf: Let C be a Hamiltonian cycle.

$F_{in} = \{ f \in F(G) : f \text{ is an inner face of } C \}$.

$E_{in} = \{ e \in E(G) : e \text{ is an inner edge of } C \}$.

By double counting of $\sum_{f \in F_{in}} d_G(f)$,

$$\sum_{f \in F_{in}} d_G(f) = |E(C)| + 2|E_{in}| \equiv 1 \pmod 2.$$

Thus there exist an odd-degree face of G.

By handshaking lemma,

$$\sum_{f \in F(G)} d_G(f) = 2|E(G)| \equiv 0 \pmod 2.$$

Thus G has an even number (≥ 2) of odd-degree faces. □

Exercise 6. A graph is called an *outerplanar graph* if it has a drawing in which every vertex lies on the boundary of the outer face. Show that a graph is outerplanar if and only if it contains no subdivision of K_4 and $K_{2,3}$.

pf:

Let H be obtained from G by adding a new vertex v adjacent to every vertex of G.

We observe G is outplanar iff H is planar.

If G is outplanar, we can place v in the outer face and join it to every vertex without crossings. Conversely take a plane drawing of H and delete v and its incident edges. The face around v merge into one face. Every remaining vertex

lies on its boundary, since every vertex was adjacent to v. Choosing the face as the outer face gives an outerplane drawing of G.

($\Rightarrow$) Assume G is outplanar. Then H is planar.
If G contains a subdivision of K_4, add 4 edges joining v to vertices corresponding to the vertices of K_4. This gives a subdivision of K_5 in H, contrary to Kuratowski's theorem

If G contains a subdivision of $K_{2,3}$, similarly we can find a subdivision of $K_{3,3}$ in H, contrary to Kuratowski's theorem.

($\Leftarrow$). Assume G has no subdivision of K_4 or $K_{2,3}$.
Suppose H is not planar, by Kuratowski's theorem,
H contains a subgraph S that is a subdivision of K_5 or $K_{3,3}$.

Call the vertices corresponding to the original vertices "branch vertices", and call the paths replacing the original edges "branch edges".

Choose a branch vertex w as follows:
① If v is a branch vertex, $w = v$.
② If v is an internal vertex of a branch path,

choose one endpoint of that path as w.

③ If $V \notin V(S)$, choose any branch vertex.

Delete w and all branch paths incident with w, keeping their other endpoints. The remaining subgraph does not contain v, so it's a subgraph of G. And it's a subdivision of K_4 or $K_{2,3}$. This contracts the assumption.

Hence H is planar, and G is outplanar. □

Exercise 7. Let G be a plane graph and let G^* be its geometric dual.

(a) Prove that G^* is a connected plane graph.

(b) Give an example showing that G need not be the geometric dual of G^*.

(c) Prove that if G is a connected plane graph, then

$$G^{**} \cong G.$$

pf: (a) By definition, there's no crossing in dual edges. Hence G^* is plane. Every two f_1^*, f_2^* can be connected by one curve which avoids $V(G)$ and cross $E(G)$ finitely.

(b) $V(G) = \{v_1, v_2\}$

$E(G) = \phi$.

$G^* = K_1$.

$G^{**} = K_1 \not\cong G$.

(c) Suppose that G_1 is connected. The dual edges crossing the edges incident with a vertex u from the boundary of a face F_u of G_1^*. Since G_1 is connected, these faces correspond one to one to the vertices of G_1. Let u^{**} be the vertex of G_1^{**} corresponding to F_u. Then

$$u \longmapsto u^{**}$$

is a bijection from $V(G_1)$ to $V(G_1^{**})$
For each edge $e = uw$ of G_1, the dual edge e^* has F_u, F_w on its two sides. Hence e^{**} joins u^{**} and w^{**}. Thus the bijection preserve the endpoints of each edge.
Hence $G_1 \cong G_1^{**}$. □

Exercise 8. Prove that no plane graph has five faces every two of which share a common boundary edge.

pf: Assume G_1 is a plane graph, and has five faces every two of which share a common boundary edge.
Thus G_1^* is a plane graph, and has five vertex every two vertices are adjacent.
Hence $K_5 \subset G_1^*$.
By Kuratowski's theorm, this contracts that G_1^* is a plane graph.
Hence the assumption is wrong,

there 's no such graph. □