Week 4 and 5

1. Prove: a connected graph G is Eulerian if and only if every edge of G is contained in an odd number of cycles.

> **Remark.** It is enough to prove that, for every edge $e = uv$, the graph $G - e$ contains an odd number of uv-paths. The technique used here is also a common counting method: Although it may be difficult to count the paths directly, we can first count the trails indirectly and then remove those trails that are not paths.

Proof. Suppose first that G is Eulerian. Let $e = uv$ be an arbitrary edge of G, and set $H = G - e$. Since every vertex of G has even degree, u and v are the only vertices of odd degree in H. Consider all trails starting at u and stopping as soon as they reach v. As long as such a trail has not reached v, it can always be extended, since the number of unused edges incident with its last vertex is odd and hence positive. Since H is finite and no edge can be used twice in a trail, the process must eventually terminate. Therefore, every maximal such trail ends at v. In the extension process above, each trail that has not yet reached v can be extended in an odd number of ways. Thus, whenever one trail is replaced by all of its extensions, one trail is replaced by an odd number of trails, so the parity of the total number does not change. Since there are an odd number of choices for the first edge at u, the total number of uv-trails obtained at the end is odd.

Now consider those uv-trails which are not paths. For each such trail, take the first repeated vertex and reverse the closed subtrail between its first two occurrences. By reversing the closed part between the first two occurrences of the repeated vertex, we obtain another different non-path uv-trail. Reversing the same part again gives back the original trail. Thus, the non-path uv-trails can be paired up, and hence their number is even. Hence, the number of such trails is even. Therefore, the number of uv-paths in H is odd.

There is a one-to-one correspondence between the uv-paths in H and the cycles of G containing e: for each uv-path in H, simply add the edge e. Hence, e is contained in an odd number of cycles. Since e was arbitrary, every edge of G is contained in an odd number of cycles.

Conversely, suppose that every edge of G is contained in an odd number of cycles. Let v be any vertex of G, $E(v)$ denote the set of edges incident with v, and let

$$E(v) = \{e_1, e_2, \ldots, e_d\},$$

where $d = d_G(v)$. For each i, let c_i be the number of cycles containing e_i. By assumption, every c_i is odd. On the other hand, every cycle containing v contains exactly two edges incident with v. Therefore,

$$\sum_{i=1}^{d} c_i = 2\big|\{C : C \text{ is a cycle containing } v\}\big|,$$

which is even. Since each c_i is odd, d is even. Since v was arbitrary, every vertex of G has even degree. As G is connected, G is Eulerian. □

2. Let G be a simple connected graph with no cut vertex and at least $n \geq 3$ vertices. Prove: if for any two vertices at distance 2,

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2} \qquad \text{(call the Fan condition)},$$

then G is Hamiltonian.

Remark. For the corresponding exercise in 徐俊明, 图论及其应用, indicate the prerequisite exercise that will be used.

First Proof. We first prove that

> Let G be a simple graph, $xy \notin E(G)$ and $d_G(x) + d_G(y) \geq n$. Then $G + xy$ is Hamiltonian if and only if G is Hamiltonian.

It suffices to prove the forward implication. The proof is similar to that of Ore's condition. Suppose that G is not Hamiltonian but $G + xy$ is Hamiltonian. Then we obtain

$$d_G(x) + d_G(y) \leq n - 1,$$

which contradicts $d_G(x) + d_G(y) \geq n$. Now let

$$S = \{v \in V(G) : d_G(v) \geq \frac{n}{2}\}.$$

According to the above conclusion, we may assume without loss of generality that $G[S]$ is a complete graph. Moreover, $G - S$ is a disjoint union of complete graphs; otherwise, there would exist two vertices at distance 2 whose degrees are both less than $n/2$, contradicting the assumption. Furthermore, distinct components of $G - S$ cannot have a common neighbor in S.

Since G is connected and has no cut vertex, there are two vertex-disjoint edges between S and each component of $G - S$. Using these edges, we can construct a Hamiltonian cycle of G. Therefore, G is Hamiltonian. □

Remark. Pósa rotation is a standard and useful technique in the study of Hamiltonian paths and cycles. Starting from a longest path, one keeps one endvertex fixed and uses an edge from the other endvertex to an internal vertex of the path to "rotate" the path and obtain a new longest path with a different endvertex. Repeating this operation often produces many possible endvertices, which can then be combined with degree conditions or connectivity assumptions to force a contradiction or to close the path into a cycle. The following solution is an example.

Second Proof. Choose a longest path

$$P = v_1 v_2 \cdots v_k$$

such that the number of its endvertices having degree at least $n/2$ is as large as possible. We first show that

$$d_G(v_1), d_G(v_k) \geq \frac{n}{2}.$$

Suppose, without loss of generality, that $d_G(v_1) < n/2$. Fix v_k and consider all longest paths obtained from P by Pósa rotations keeping v_k fixed. By the choice of P, every possible other endvertex x of such a path satisfies

$$d_G(x) < \frac{n}{2}. \tag{1}$$

Let

$$x_1x_2\cdots x_k, \qquad x_k = v_k,$$

be one of these paths, and let

$$m = \max\{i : x_1x_i \in E(G)\}.$$

We claim that

$$x_1x_i \in E(G) \qquad \text{for all } 2 \le i \le m. \tag{2}$$

Indeed, otherwise there is some i such that

$$x_1x_i \notin E(G), \qquad x_1x_{i+1} \in E(G).$$

Rotating the path through the edge x_1x_{i+1} gives another longest path with endvertex x_i. Hence, by (1),

$$d_G(x_1) < \frac{n}{2}, \qquad d_G(x_i) < \frac{n}{2}.$$

But $d_G(x_1, x_i) = 2$, contradicting the Fan condition. Thus (2) holds.

Now choose such a rotated path $P = v_1\cdots v_k$ so that

$$m = \max\{i : v_1v_i \in E(G)\}$$

is as large as possible. By (2),

$$v_1v_i \in E(G) \qquad (2 \le i \le m). \tag{3}$$

For every $j < m$, rotating P through v_1v_{j+1} gives a longest path with endvertex v_j. Hence v_j has no neighbor outside $V(P)$. Moreover, v_j has no neighbor v_s with $s > m$; otherwise the same rotation would give a longest path whose new endvertex is adjacent to v_s, contradicting the maximality of m. Therefore

$$N_G(v_j) \subseteq \{v_1, \ldots, v_m\} \qquad (1 \le j < m). \tag{4}$$

Since G has no cut vertex, (4) is impossible unless $m = k$. Thus $v_1v_k \in E(G)$, and P can be closed into a cycle containing all vertices of P. Since P is a longest path and G is connected, this cycle must contain all vertices of G. Hence G is Hamiltonian. □

3. Prove the following statement (Proposition 12.5): the line graph $L(G)$ of any graph G is claw-free.

Proof. Let $e = uv \in E(G)$. The neighbors of e in $L(G)$ correspond to the edges of G incident with either u or v. Consider any three neighbors of e in $L(G)$. Among the three corresponding edges of G, at least two are incident with the same endpoint of e. Hence, these two edges are adjacent in $L(G)$. Therefore, no three neighbors of e are pairwise non-adjacent. Since e was arbitrary, $L(G)$ is claw-free. □

4. Let x, y, z be three distinct vertices of a strongly connected digraph D. Prove:

$$d_D(x, z) \le d_D(x, y) + d_D(y, z).$$

Proof. Let P_1 be a shortest directed xy-path in D, and let P_2 be a shortest directed yz-path in D. Then the lengths of P_1 and P_2 are $d_D(x, y)$ and $d_D(y, z)$, respectively.

By concatenating P_1 and P_2, we obtain a directed xz-walk of length

$$d_D(x, y) + d_D(y, z).$$

If this walk contains a repeated vertex, we can delete the closed part between two occurrences of this vertex. Repeating this process, we obtain a directed xz-path whose length is no greater than the length of the original walk. Therefore,

$$d_D(x,z) \leq d_D(x,y) + d_D(y,z).$$

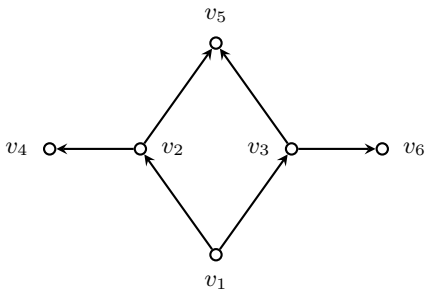
□

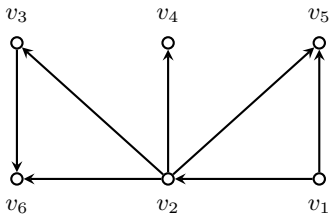
5. Let A be the adjacency matrix of a (directed or undirected) graph D, and I the $n \times n$ identity matrix. The polynomial

$$P_D(\lambda) = \det(\lambda I - A) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n$$

is called the *characterstic polynomial* of D; the roots of $P_D(\lambda)$ are called the *characterstic roots* of D. The characterstic roots together with their multiplicities from the spectrum.

(a) Find the characterstic polynomial (both are of degree 6) and characterstic roots of the following two graphs:





Proof. Label the vertices as shown in the figure. For the first digraph, the adjacency matrix is

$$A_1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\lambda I - A_1 = \begin{pmatrix} \lambda & -1 & -1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -1 & -1 & 0 \\ 0 & 0 & \lambda & 0 & -1 & -1 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Since this matrix is upper triangular,

$$P_{D_1}(\lambda) = \det(\lambda I - A_1) = \lambda^6.$$

Hence the only characteristic root of D_1 is 0, with multiplicity 6.

For the second digraph, the adjacency matrix is

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore,

$$\lambda I - A_2 = \begin{pmatrix} \lambda & -1 & 0 & 0 & -1 & 0 \\ 0 & \lambda & -1 & -1 & -1 & -1 \\ 0 & 0 & \lambda & 0 & 0 & -1 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Again, this matrix is upper triangular, so

$$P_{D_2}(\lambda) = \det(\lambda I - A_2) = \lambda^6.$$

Hence the only characteristic root of D_2 is also 0, with multiplicity 6.

Thus, both digraphs have characteristic polynomial

$$\lambda^6,$$

and both have 0 as their only characteristic root, with multiplicity 6. □

Remark. It suffices to consider a topological ordering of the digraph. A topological ordering is an ordering $v_1, v_2, \ldots, v_n$ of the vertices such that every arc $v_i v_j$ satisfies $i < j$. Since the digraph is acyclic, such an ordering exists, and with respect to this ordering, its adjacency matrix is strictly upper triangular.