– Graph Theory and Its Applications –

Plane Graphs and Planar Graphs

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1 Plane Graphs and Euler's Formula

- Let S be a given surface, such as the plane, the sphere, the double torus, and so on. A graph D is said to be embeddable in the surface S if it can be drawn on S so that its edges meet only at their end-vertices.
- Such a drawing $\widetilde{D}$ of D on S is called a representation of D on S.
- Whether a graph can be embedded in a surface S does not depend on the directions of its edges, so in what follows we only need to consider embeddings of undirected graphs G.
- This chapter only deals with the case where S is the plane or the sphere. In fact, we have the following statement.

Theorem 3.1.1 A graph G can be embedded in the sphere S if and only if G can be embedded in the plane P.

Proof

- Consider the stereographic projection (see Figure 1). Let the sphere S be tangent to the plane P, and let z be the point of S diametrically opposite the point of tangency. Define a map $\varphi : S \to P$ as follows: for any point $s \in S \setminus \{z\}$ and $p \in P \setminus \{\infty\}$, set $\varphi(s) = p$ precisely when z, p, s are collinear, and put $\varphi(z) = \infty$. It is easy to see that φ is a bijection.

- Suppose G can be embedded in the plane P, and let $\widetilde{G}$ be a representation of G on P. Then $\varphi^{-1}(\widetilde{G})$ is a representation of G on the sphere S.

- Conversely, let $\widetilde{G'}$ be a representation of G on S. We may assume that z lies neither on a vertex nor on an edge of $\widetilde{G'}$; then $\varphi(\widetilde{G'})$ is a representation of G on P. Therefore G is embeddable in the sphere S if and only if it is embeddable in the plane P. ∎

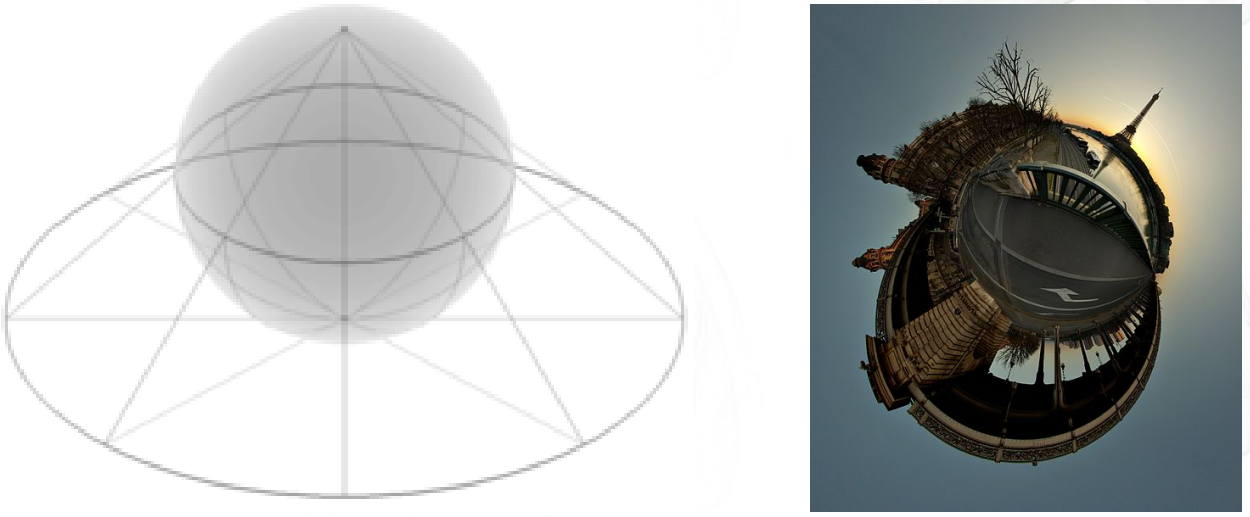


Figure 1: The stereographic projection

- A graph G that can be embedded in the plane (or the sphere) is called a planar graph.
- A graph that cannot be embedded in the plane (or the sphere) is called a non-planar graph.
- A plane representation of G is denoted by $\widetilde{G}$. The drawing $\widetilde{G}$ itself may be regarded as a graph isomorphic to G; it is called a plane

graph.

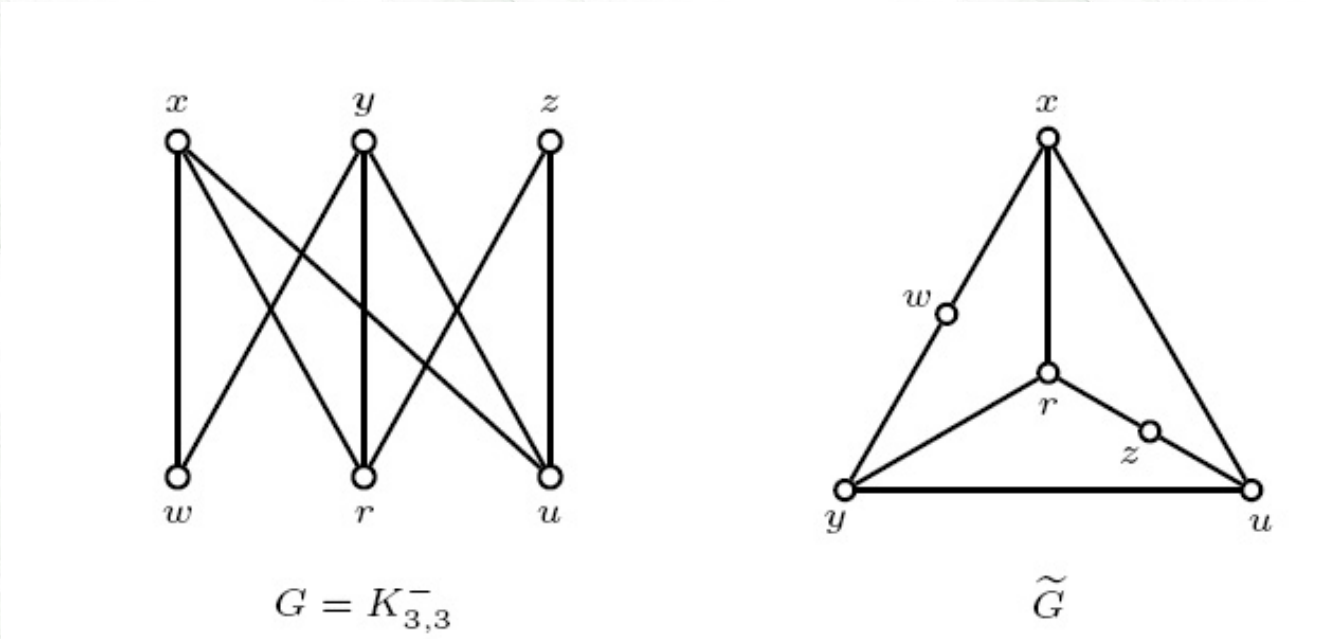


Figure 2: A planar graph G and its plane representation $\widetilde{G}$

- Figure 2 shows a planar graph G and its plane representation $\widetilde{G}$, where $G = K_{3,3}^{-}$ (the graph obtained from $K_{3,3}$ by deleting one edge;

here $G = K_{3,3} - zw$). This plane representation of $K_{3,3}^-$ may also be viewed as an embedding on a tetrahedron, with all its vertices and edges lying on the edges of the tetrahedron. This representation will be used in the proof of Theorem 3.2.

- Intuitively, a plane representation $\widetilde{G}$ of a nonempty planar graph divides the plane P into several connected regions; these regions are called the faces of G.

- In every plane representation $\widetilde{G}$ of a planar graph G, there is exactly one face containing the point ∞ of the plane P; it is called the exterior face of $\widetilde{G}$.

- A face that does not contain the point ∞ of the plane P is called an interior face of $\widetilde{G}$.

- Let G be a planar graph and let x be any vertex of G. It is not hard to show (Exercise 3.1.2) that there is a plane representation $\widetilde{G}$ of G in which x lies on the boundary of the exterior face of $\widetilde{G}$.

- We write $F(G)$ and $\phi(G)$ for the set of faces and the number of faces of the plane graph G, respectively.
- It is easy to see that for every plane graph G, $\phi(G) \geqslant 1$; moreover, $\phi(G) = 1$ if and only if G is a forest, and $\phi(G) \geqslant 2$ if and only if G contains a cycle.
- The closed walk formed by the edges surrounding a face f is called the boundary of f. We write $B_G(f)$ for the boundary of the face f in the plane graph G.
- The number of edges in $B_G(f)$ is denoted by $d_G(f)$ (a bridge is counted twice) and is called the degree of the face f (think of it as the number of edges traversed by a person who starts at any vertex on the boundary of f, walks once around that boundary, and returns to the starting point).
- For example, in the plane graph G shown in Figure 3, the boundary

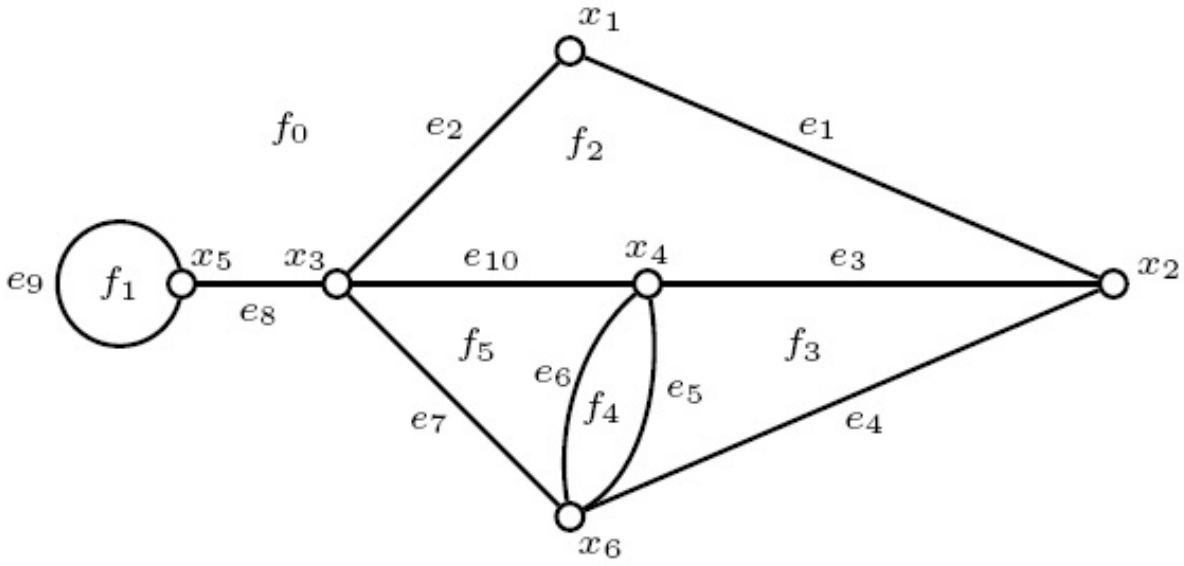


Figure 3: Faces of a plane graph

of the face f_0 is

$$B_G(f_0) = x_3\ e_2\ x_1\ e_1\ x_2\ e_4\ x_6\ e_7\ x_3\ e_8\ x_5\ e_9\ x_5\ e_8\ x_3.$$

We have $d_G(f_0) = 7$ and $d_G(f_1) = 1$.

- It is easy to see that the boundary of every interior face of a connected plane graph G consists of a single closed walk of G. The boundary of the exterior face of a connected plane graph likewise consists of a single closed walk; for a disconnected plane graph it consists of several vertex-disjoint closed walks.

The face degrees $d_G(f)$ of a plane graph G and its number of edges $e(G)$ satisfy the following relation.

Theorem 3.1.2 For every plane graph G,

$$\sum_{f \in F(G)} d_G(f) = 2e(G).$$

Proof

- If G is the empty graph, the statement holds trivially.

- Now suppose G is nonempty, and let e be any edge of G. Then e either lies on the common boundary of two faces (as the edge e_1 of G in Figure 3, which lies on the common boundary of f_0 and f_2), or appears twice on the boundary of a single face (as the edge e_8 in the example above, which appears twice on the boundary of f_0).
- Hence

$$\sum_{f \in F(G)} d_G(f) = 2e(G).$$

This completes the proof. ∎

Euler's Formula

Let G be a connected plane graph, and let n, m and ϕ denote its numbers of vertices, edges and faces. It is believed that in 1750 Euler obtained the celebrated Euler polyhedron formula (also called the planar graph formula).

Lemma 1 *Let G be a connected plane graph and let $e \in E(G)$.*

(i) If e lies on some cycle C of G, then e lies on the boundaries of exactly two faces of G;

(ii) if e is a bridge of G, then e lies on the boundary of exactly one face of G.

Theorem 3.1.3 (Euler's formula, Euler, 1753) If G is a connected plane graph, then

$$n + \phi - m = 2 \quad (\text{or } |V(G)| + |F(G)| - |E(G)| = 2). \tag{1}$$

Proof: Method 1

- Let G be a connected plane graph and let T be a spanning tree of G. Then $\phi(T) = 1$ and $e(\overline{T}) = m - n + 1$.

- By Lemma 1, each deletion of an edge of the cotree $\overline{T}$ from G reduces the number of faces by one. Thus

$$\phi(G) - e(\overline{T}) = \phi(G) - (m - n + 1) \geqslant \phi(T). \tag{2}$$

- On the other hand, each edge of $\overline{T}$ added back to T creates one new face of G. Hence

$$\phi(G) \leqslant \phi(T) + e(\overline{T}) = \phi(T) + m - n + 1. \tag{3}$$

- From (2) and (3) we obtain

$$\phi(G) = \phi(T) + m - n + 1 = m - n + 2,$$

which is Euler's formula (1).

Method 2

- We use induction on $\phi \geqslant 1$.

- For $\phi = 1$, G is a tree with $m = n - 1$, and the assertion is obvious. Assume the assertion holds for every connected plane graph with fewer than k faces.
- Let G be a connected plane graph with $\phi(G) = k \geqslant 2$. Then G contains a cycle C.
- Choose any $e \in E(C)$; then $G - e$ is a connected plane graph with $\phi(G - e) = k - 1$.
- By the induction hypothesis,

$$n(G-e) - e(G-e) + \phi(G-e) = 2.$$

- Using

$$n(G-e) = n, \qquad e(G-e) = e(G) - 1,$$

$$\phi(G-e) = \phi(G) - 1,$$

- we obtain

$$n - m + \phi(G) = 2.$$

This completes the proof. ∎

Corollary 3.1.3.1 If G is a plane graph, then $n + \phi - m = 1 + \omega$.

Corollary 3.1.3.2 All plane representations $\widetilde{G}$ of a planar graph G have the same number of faces.

Corollary 3.1.3.3 If G is a simple bipartite connected planar graph of order $n (\geqslant 3)$, then $e(G) \leqslant 2n - 4$.

Proof

- Since $\widetilde{G}$ is bipartite, it contains no odd cycle. Hence for every $f \in F(\widetilde{G})$ we have $d_{\widetilde{G}}(f) \geqslant 4$.

- By Theorem 3.1.2,

$$4\phi \leqslant \sum_{f \in F(\widetilde{G})} d_{\widetilde{G}}(f) = 2e(G),$$

that is, $e(G) \geqslant 2\phi$.

- Euler's formula (1) then gives $e(G) \leqslant 2n-4$. ∎

Corollary 3.1.3.4 $K_{3,3}$ is a non-planar graph.

- Let G be a simple planar graph and let x, y be two nonadjacent vertices of G. If $G+xy$ is non-planar, then G is called a maximal planar graph.
- It follows immediately from the definition that in every plane representation $\widetilde{G}$ of a maximal planar graph G of order $n(\geqslant 3)$, every face is a triangle.

- For instance, Figure 4(a) shows a maximal planar graph, and (b) shows one of its plane representations.

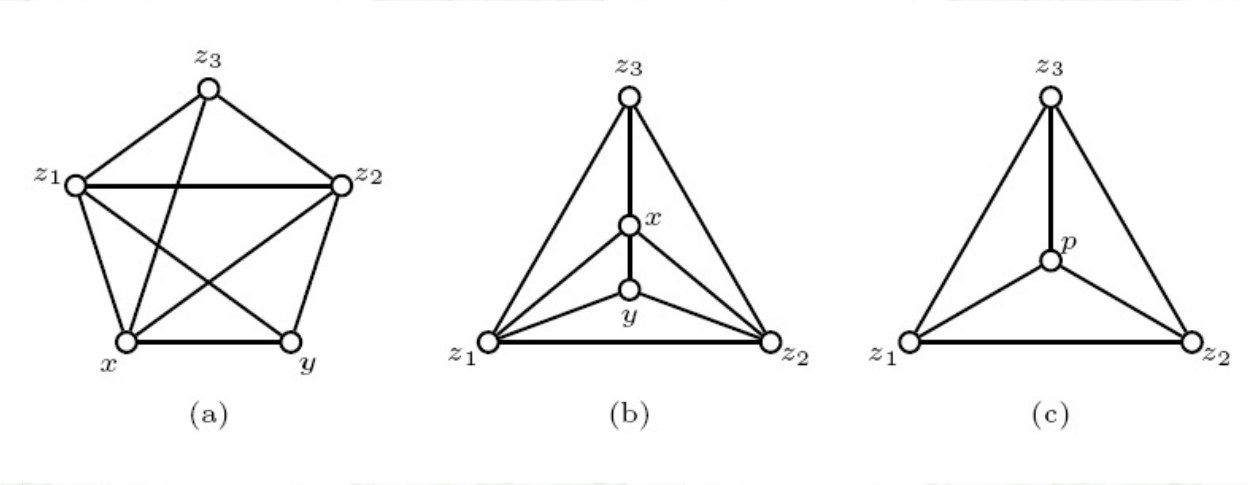


Figure 4: Auxiliary figures for the proof of Theorem 3.1.5

- A plane representation of a maximal planar graph is called a triangulation of a plane graph, or simply a triangulation.

Corollary 3.1.4.1 If G is a simple planar graph of order $n(\geqslant 3)$, then $e(G) \leqslant 3n-6$. ∎

Corollary 3.1.4.2 If G is a simple planar graph, then $\delta \leqslant 5$.

Proof

- For $n = 1, 2$ the assertion is obvious.
- For $n \geqslant 3$, Corollary 1.3.1 and Corollary 3.1.4.1 give

$$\delta n \leqslant \sum_{x \in V} d_G(x) = 2e(G) \leqslant 2(3n-6),$$

hence $\delta \leqslant 5$. ∎

Corollary 3.1.4.3 K_5 is a non-planar graph.

Proof This follows by directly counting its edges. ∎

The following characterization of planar graphs was discovered independently by K. Wagner (1936) and I. Fáry (1948).

Theorem 3.1.5 Every simple planar graph G has a plane representation $\widetilde{G}$ in which every edge is a straight-line segment.

Proof

- It suffices to prove the theorem for maximal planar graphs.
- We induct on the order $n(\geqslant 3)$. For $n = 3$, G is a triangle and the statement is obvious.
- Assume the statement holds for every maximal planar graph of order less than n, and let G be a maximal planar graph of order $n(\geqslant 4)$. We may assume that G is a triangulation.
- Choose $x \in V(G)$ that is not on the boundary of the exterior face, and take $y \in N_G(x)$. Then xy is the common edge of two interior triangles.

- Let these two triangles be (z_1, x, y, z_1) and (z_2, x, y, z_2) (see Figure 4(b)).
- Consider the graph $G \cdot xy$ obtained by contracting the edge xy, with x and y identified to a vertex p, and let G' be the graph obtained from $G \cdot xy$ by deleting parallel edges (see Figure 4(c)).
- Clearly G' is planar, and by Theorem 3.1.4,

$$e(G') = e(G) - 3 = 3\,(n(G) - 1) - 6 = 3\,n(G') - 6.$$

- Again by Theorem 3.1.4, G' is a maximal planar graph of order $n-1$.
- By the induction hypothesis, G' has a plane representation $\widetilde{G}'$ in which every edge is a straight segment.
- Split the edges pz_1 and pz_2 in $\widetilde{G}'$ so as to form two triangles (see Figure 4(c),(b)). The resulting drawing $\widetilde{G}$ is a plane representation

of G in which every edge is a straight segment. By induction, the theorem follows. ∎

Exercises

3.1.4 Let G be a graph of order $n\,(\geqslant 4)$, let $\Delta = \Delta(G)$, and let n_i denote the number of vertices of degree i in G. Prove that

(a) if G is a triangulation, then $3n_3 + 2n_4 + n_5 = n_7 + 2n_8 + \cdots + (\Delta - 6)n_\Delta + 12$;

(b) if G is a tree, then $n_1 = n_3 + 2n_4 + 3n_5 + \cdots + (\Delta - 2)n_\Delta + 2$.

3.1.7 Prove that (a) if G is a connected planar graph of order n and girth $g \geqslant 3$, then $e(G) \leqslant g(n-2)/(g-2)$;

(b) the Petersen graph is non-planar.

3.1.10 Let G be a simple planar graph with $\Delta(G) = n - 1$. Prove that if $n \geqslant 5$, then G contains two nonadjacent vertices both of degree at most 3.

2 Kuratowski's Theorem

- Let $e \in E(G)$, $e = xy$. An edge subdivision is the operation of deleting e from G and joining x and y by a path of length 2 whose internal vertex is disjoint from G.
- A subdivision of a graph G is a graph obtained by carrying out a sequence of edge subdivisions on G.
- For example, Figure 5 shows the subdivision of one edge of K_5 and a subdivision of $K_{3,3}$.

- Let H be a subgraph of G. Define a relation "$\sim$" on $E(G) \setminus E(H)$ as follows: $e_1 \sim e_2$ if and only if there is a trail W in $G - E(H)$ such that
  (i) the first and last edges of W are e_1 and e_2, respectively;
  (ii) W is internally vertex-disjoint from H.

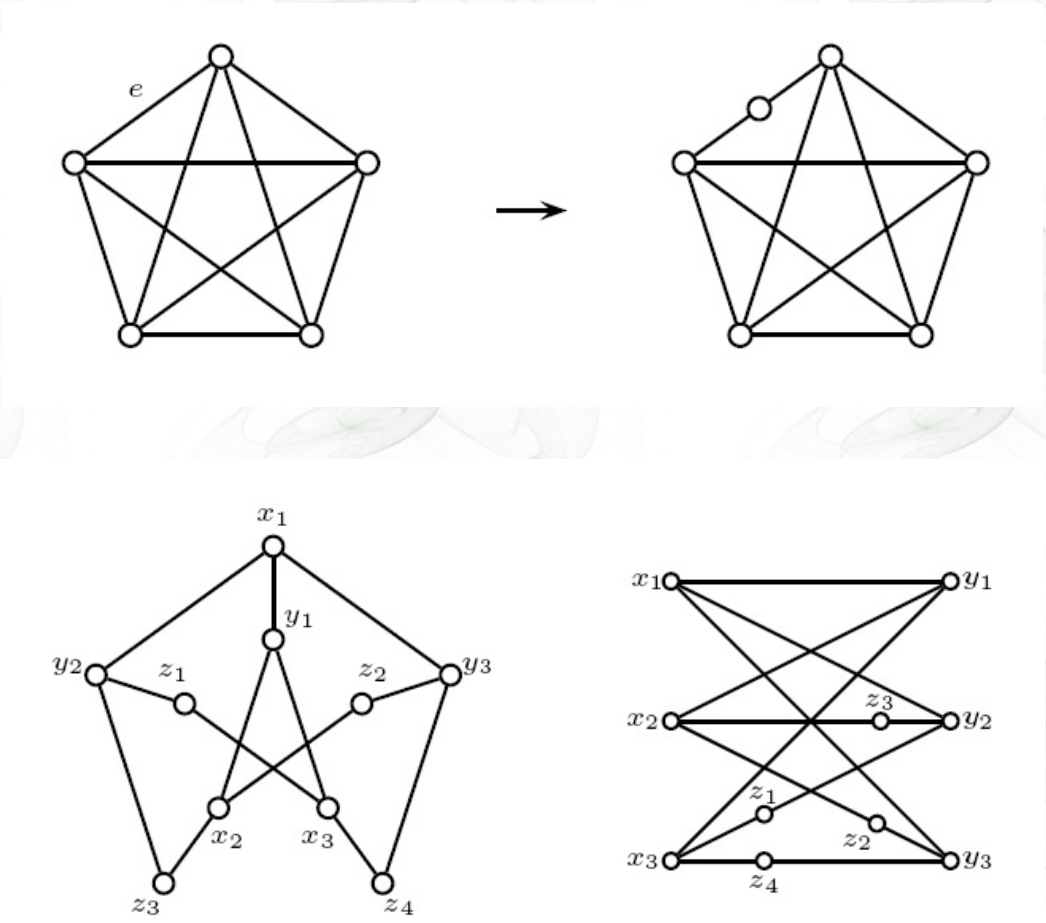


Figure 5: The subdivision of an edge e of K_5, and a subdivision of $K_{3,3}$

- It is easy to check that "$\sim$" is an equivalence relation on $E(G) \setminus E(H)$. The subgraphs of $G - E(H)$ induced by the equivalence classes of "$\sim$" are called the H-bridges of G.

- It follows directly from the definition that if B is an H-bridge of G, then B is connected, any two vertices of B are joined by a path internally vertex-disjoint from H, and any two distinct H-bridges are edge-disjoint.

- For an H-bridge B, write $V_G(B,H) = V(B) \cap V(H)$ and call $V_G(B,H)$ the set of vertices of attachment of B to H in G.

- In the graph G of Figure 6, the subgraph H is the cycle $(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_1)$; B_1,B_2,B_3,B_4 are four H-bridges, with $V_G(B_1,H) = V_G(B_2,H) = \{x_1,x_2,x_3\}$, $V_G(B_3,H) = \{x_4,x_6,x_7,x_9\}$ and $V_G(B_4,H) = \{x_5,x_8\}$.

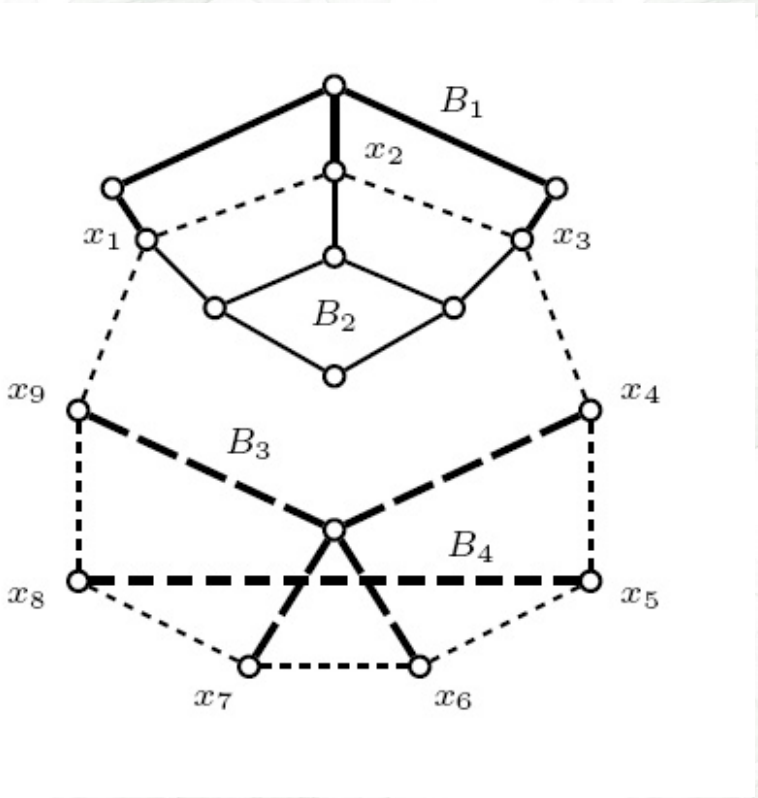


Figure 6: The H-bridges of G

The following theorem gives a characterization of planar graphs. It was discovered independently by the Polish mathematician C. Kuratowski (1930) and by O. Frink and P. A. Smith (1930), and is generally

known in the literature as Kuratowski's theorem. The proof given here is due to H. Tverberg (1989).

Theorem 3.2 (Kuratowski's theorem) A graph G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Proof

- The necessity is immediate.
- We now prove sufficiency. Clearly it is enough to consider connected simple graphs. For graphs of order $n \leqslant 5$, sufficiency is easily verified.
- (Proof by contradiction.) Suppose G is a non-planar simple graph of minimum order $n\,(\geqslant 6)$ and maximum size containing no subdivision of K_5 or $K_{3,3}$. Then G has no cut-vertex and contains a subdivision of K_5^- (the graph obtained from K_5 by deleting one edge) or of $K_{3,3}^-$ (obtained from $K_{3,3}$ by deleting one edge); denote this subdivision by H.

- Claim: G must contain a subdivision of $K_{3,3}^-$.
- Suppose first that H is a subdivision of K_5^-. Since H is planar and G is non-planar, there must be an H-bridge B of G having at least two vertices of attachment x and y to H (because G has no cut-vertex).
- Let P be an xy-path in B (shown by the thick edges in Figure 7(a)); then $xy \in E(H)$ (otherwise G would contain a subdivision of K_5).
- Since G is simple, $n(P) \geqslant 3$, and $H \cup P$ contains a subdivision of $K_{3,3}^-$ (see Figure 7(b)). This proves the claim.
- Thus we may assume that H is a subdivision of $K_{3,3}^-$ with bipartition $\{X,Y\}$, where $X = \{x,y,z\}$ and $Y = \{r,u,w\}$, and with the "edge" P_{zw} missing (see Figure 7(b)).
- Since planar and spherical embeddings are equivalent, we seek a contradiction by trying to embed G on the surface of a tetrahedron.

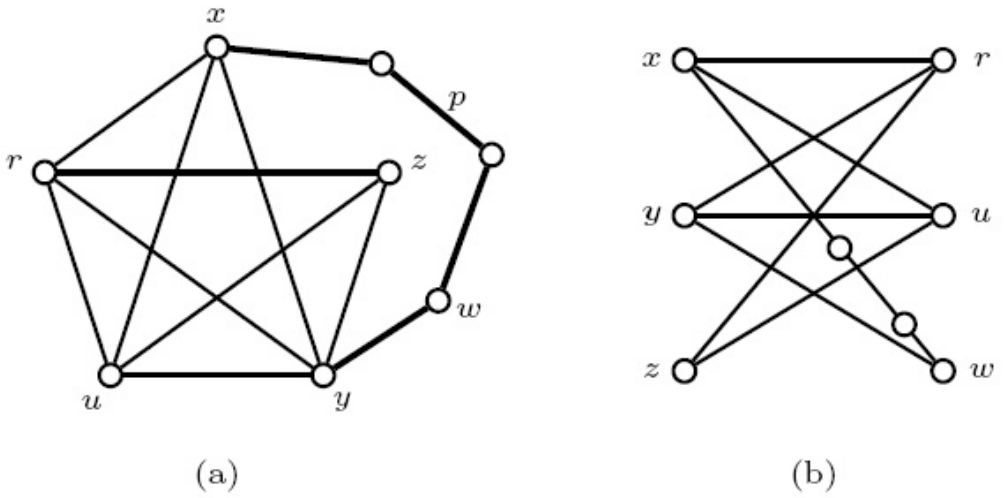


Figure 7: (a) The subgraph $H \cup P$ of G; (b) a subdivision of $K_{3,3}^{-}$ contained in $H \cup P$

- Let the four vertices of a tetrahedron T be x, y, r, u; all vertices of the planar graph H other than x, y, r, u are embedded on the edges of T.
- Hence H has at least two further vertices, say w and z, embedded on

two skew edges of T. We may assume that w and z are embedded on the edges xy and ru of T, respectively (see Figure 8).

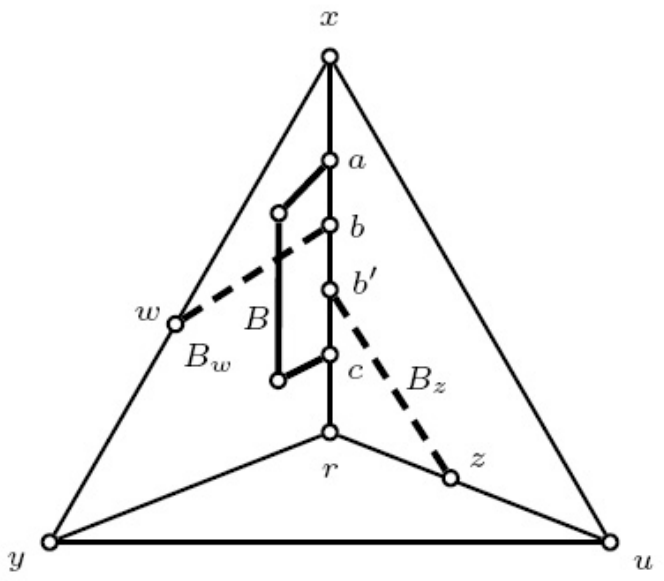


Figure 8: Auxiliary figure for the proof of Theorem 3.2

- By the non-planarity of G and the minimality of $n(G)$, we have

$d_G(w) \geqslant 3$ and $d_G(z) \geqslant 3$, so both w and z are vertices of attachment of some H-bridges.

- Let B be an H-bridge. Since G contains no subdivision of $K_{3,3}$, the vertices of attachment of B to H cannot lie on two skew edges of T (otherwise, for example if w and z both lay on B, then G would contain a subdivision of $K_{3,3}$).
- Hence the vertices of attachment of B to H either all lie on a single edge of T (in which case B is called a small bridge), or they lie on two or three boundary edges of a single face of T (in which case B is called a large bridge). Thus every bridge of G lies on some one face of T.
- Observe that the embeddings of distinct large bridges do not interfere with one another, and neither do those of small bridges on distinct edges.
- Let B_w and B_z be the H-bridges containing the vertices w and z, respectively.

- By the minimality of $n(G)$, both $G-w$ and $G-z$ can be embedded on T. Hence, if B_w is a small bridge, it does not affect the embedding of the remaining bridges on T. Therefore, if G cannot be embedded on T, then B_w must be a large bridge; similarly B_z is a large bridge.

- But since neither B_w nor B_z interferes with the embedding of the other large bridges, B_w and B_z must interfere with the embedding of some small bridge B on T.

- We may assume that the vertices of attachment of B to H lie on the edge xr of T. Then on the edge xr there are points a,b,b',c, where b,b' (possibly $b'=b$) lie between a and c, a and c are vertices of attachment of B to H, and b and b' are the vertices of attachment of B_w and B_z to H, respectively (see Figure 8).

- Replace the ac-path of H along the edge xr of T by the ac-path in B (shown by the thick edges in Figure 8), and denote the resulting subgraph by H'. Let P be the wz-path obtained as the union of

the wb-path in B_w, the bb'-path along the edge xr of T, and the $b'z$-path in B_z (shown by the thick dashed edges in Figure 8). Then $H' \cup P$ is a subdivision of $K_{3,3}$, contradicting the hypothesis. This proves sufficiency. ∎

From Kuratowski's theorem (3.2) we immediately conclude that every tree is planar. We can also conclude that the Petersen graph is non-planar, since it contains a subdivision of $K_{3,3}$, as shown in Figure 5, with bipartition $\{X,Y\}$, $X = \{x_1,x_2,x_3\}$ and $Y = \{y_1,y_2,y_3\}$.

There are several other characterizations of planar graphs. For instance, K. Wagner (1937) proved: a graph G is planar if and only if it contains no subgraph contractible to K_5 or $K_{3,3}$.

Likewise, S. MacLane (1937) proved: a graph G is a bridgeless planar graph if and only if it has a cycle basis (a set of cycles generating its

whole cycle space) such that the union of these cycles with any further cycle contains every edge of G exactly twice.

One of the famous conjectures in structural graph theory is the Cycle Double Cover Conjecture.

Conjecture 2.1 (Szekeres, 1973, and Seymour, 1980) *Every bridgeless graph has a multiset of cycles covering every edge exactly twice. (Solved by GPT this year!!!)*

Exercises

3.2.1 Prove that (a) if G is non-planar, then every subdivision of G is non-planar;

(b) if G is planar, then G contains no subdivision of K_5 or $K_{3,3}$;

(c) if G is planar, then every subgraph of G is planar.

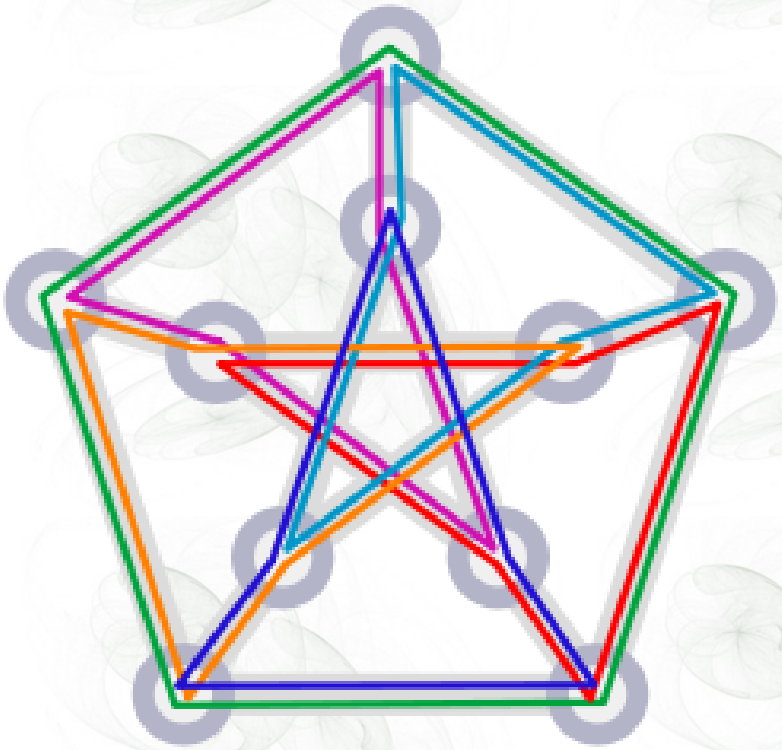


Figure 9: A 5-cycle double cover of the Petersen graph

3.2.6 Let G be a plane graph of odd order. Prove that if G has a Hamiltonian cycle, then G has an even number $(\geqslant 2)$ of odd-degree faces.

3.2.7 A graph is called an outerplanar graph if it has a drawing in which

every vertex lies on the boundary of the outer face. Show that a graph is outerplanar if and only if it contains no subdivision of K_4 and $K_{2,3}$.

3 Dual Graphs

- Let G be a plane graph with $F(G) = \{f_1, f_2, \cdots, f_\phi\}$. Define a graph G^* as follows: to each face f_i of G there corresponds a vertex f_i^* of G^*, so that $V(G^*) = \{f_1^*, f_2^*, \cdots, f_\phi^*\}$, and to each edge e of G there corresponds an edge e^* of G^*, so that $E(G^*) = \{e^* \mid e \in E(G)\}$, where f_i^* and f_j^* are joined by e^* if and only if the faces f_i and f_j of G share the boundary edge e. The graph G^* is called the geometric dual graph of G.

- Figure 10 shows a plane graph G (thin lines) and its geometric dual G^* (thick lines).

- It is easy to see that the geometric dual G^* of a plane graph G is again a plane graph (Exercise 3.3.2).

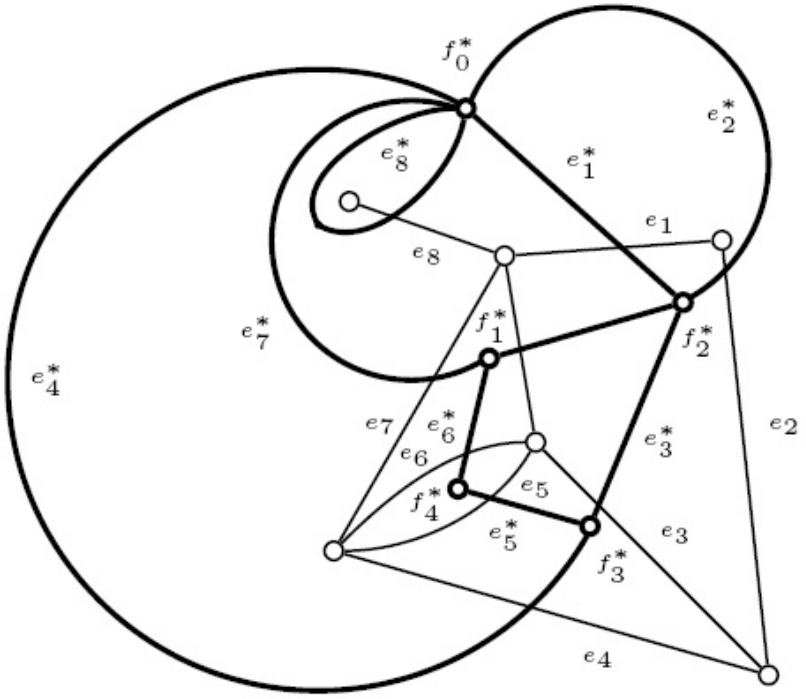


Figure 10: The geometric dual G^* (thick lines) of a plane graph G (thin lines)

- It is worth noting that isomorphic plane graphs may have non-

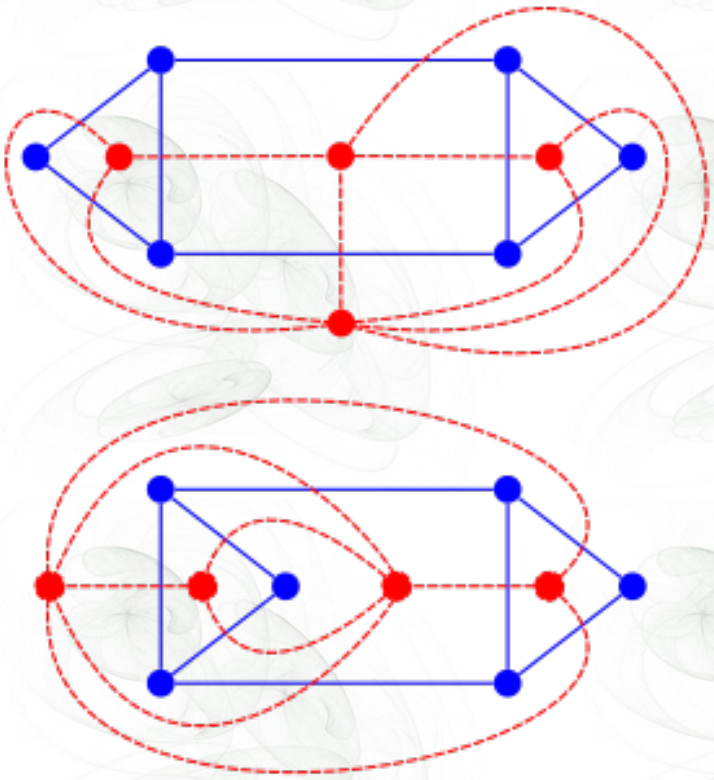


Figure 11: Two isomorphic plane graphs whose geometric duals are not isomorphic

isomorphic geometric duals. For instance, the two plane graphs in Figure 11 are isomorphic, but their geometric duals are not: the

graph on the left has a face of degree 5, whereas the graph on the right has no such face.

- Therefore the notion of a geometric dual is meaningful only for plane graphs and in general cannot be extended to planar graphs.
- From the definition of G^* we immediately have

$$\begin{cases} n(G^*) = \phi(G), & e(G^*) = e(G), \\ d_{G^*}(f^*) = d_G(f), & \forall\ f \in F(G). \end{cases} \tag{4}$$

Theorem 3.3.1 Let G^* be the geometric dual of a plane graph G, let $B \subseteq E(G)$, and put $B^* = \{e^* \in E(G^*) : e \in B\}$. Then

(i) $G[B]$ is a cycle of G if and only if B^* is a bond of G^*;

(ii) B is a bond of G if and only if $G^*[B^*]$ is a cycle of G^*.

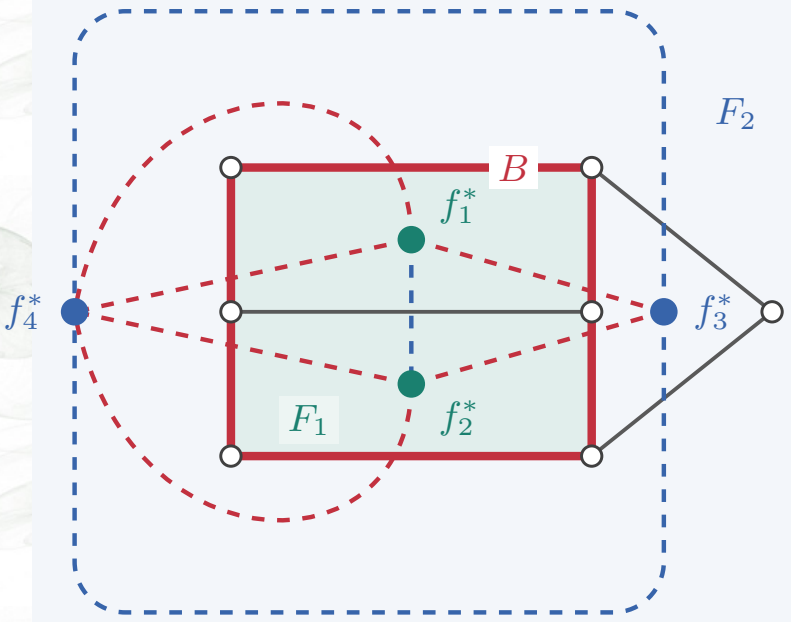
Proof (i) ($\Leftarrow$)

- Suppose B^* is a bond of G^*; we prove that $G[B]$ is a cycle of G.
- Since B^* is a bond of G^*, there are two disjoint nonempty proper subsets F_1^* and F_2^* of $V(G^*)$ such that $B^* = E_{G^*}[F_1^*, F_2^*]$.
- Since $V(G^*) = F(G)$, $G[B]$ partitions $F(G)$ into two disjoint nonempty proper subsets F_1 and F_2, where F_i $(i = 1,2)$ is the set of faces of G corresponding to F_i^*.
- Because B^* is a minimal edge cut, $G^*[F_i^*]$ is connected.
- Let G_i be the union of the faces in F_i $(i = 1,2)$. Since $F_i \neq \emptyset$, G_1 has a boundary H with $E(H) = B$.
- Since $G^*[F_1^*]$ is connected, the exterior-face boundary $H = G[B]$ of G_1 is connected.
- Moreover, every edge of H is the common boundary of a face in G_1 and a face in G_2, so every vertex of $H = G[B]$ has even degree; hence Hcontains a cycle of G.

- By the minimality of $G^*[B^*]$, we have $G[B]$ is a cycle.

A cycle in $G \longleftrightarrow$ a bond in G^*

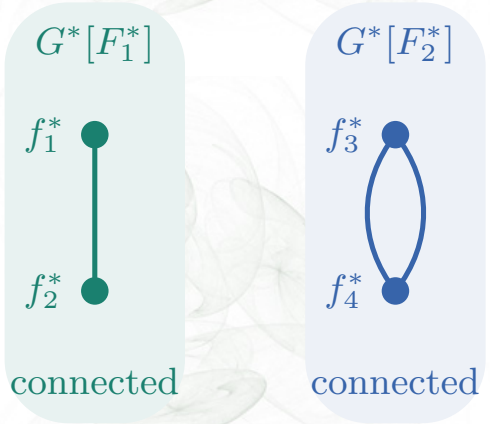
(a) G and its geometric dual G^*



$B^* = \delta_{G^*}(F_1^*)$

(b) Delete the six red dual edges

$G^* - B^*$



Every edge of B^* joins the two parts.
Restoring any one reconnects them.

B^* is a bond

B in G B^* in G^* other dual edges

($\Rightarrow$)

- Suppose $G[B]$ is a cycle of G; we prove that B^* is a bond of G^*.
- Since $G[B]$ is a cycle, it partitions the face set $F(G)$ of G into two disjoint nonempty proper subsets F_1 and F_2.
- Let F_i^* $(i = 1,2)$ be the vertex set of G^* corresponding to F_i.
- We may assume that F_1 lies inside $G[B]$. Then $G[B]$ is the exterior-face boundary of the union of the faces in F_1; that is, every edge of B lies on the common boundary of some face in F_1 and some face in F_2.
- In other words, every edge e of B lies on the common boundary of the interior and exterior faces of $G[B]$, so e^* joins a vertex of F_1^* to a vertex of F_2^*; thus B^* is the edge cut $E_{G^*}(F_1^*, F_2^*)$ of G^*.
- Since both $G^*[F_1^*]$ and $G^*[F_2^*]$ are connected, Exercise 2.2.3 shows that $B^* = E_{G^*}[F_1^*, F_2^*]$ is a bond of G^*.

The proof of (ii) is similar to that of (i) and is left to the reader (Exercise 3.3.4). ∎

Corollary 3.3.1.1 Let T be a spanning tree of a connected plane graph G, and let $E^* = \{e^* \in E(G^*) : e \in E(\overline{T})\}$. Then $T^* = G^*[E^*]$ is a spanning tree of G^*.

Proof

- When $\phi(G) = 1$, we have $G = T$, $E^* = \emptyset$, and both G^* and T^* are trivial graphs, so the statement holds.
- Now suppose $\phi(G) \geqslant 2$. Since the boundary of every face of G contains an edge of $E(\overline{T})$, every vertex of G^* is incident with an edge of E^*; hence T^* is a spanning subgraph of G^*.
- If T^* contained a cycle of G^*, then by Theorem 3.3.1 the set $E(\overline{T})$ would contain a bond of G, contradicting Theorem 2.1.6(i). Thus T^* is acyclic.

- $$e(T^*) = e(\overline{T}) = e(G) - n + 1 = \phi - 2 + 1 = n(G^*) - 1,$$

  where the third equality uses Euler's formula. By Theorem 2.2.3, T^* is a spanning tree of G^*. ∎

Corollary 3.3.1.2 If G is a connected plane graph, then

$$\dim \mathcal{C}(G^*) = \dim \mathcal{B}(G),$$

$$\dim \mathcal{B}(G^*) = \dim \mathcal{C}(G).$$

Combinatorial dual graphs

For interested students; optional self-study.

Exercises

3.3.1 Let G be a plane graph and let G^* be its geometric dual.

(a) Prove that G^* is a connected plane graph.

(b) Give an example showing that G need not be the geometric dual of G^*.

(c) Prove that if G is a connected plane graph, then $G^{**} \cong G$.

3.3.6 Prove that no plane graph has five faces every two of which share a common boundary edge.

4 Regular Polyhedra

- The theory of plane graphs is closely connected with the study of convex polyhedra; indeed, every convex polyhedron corresponds to a plane graph.
- Let P be a convex polyhedron. The plane graph $G(P)$ whose vertices and edges are the vertices and edges of P is called the plane graph corresponding to P.
- It is easy to see that $G(P)$ is connected with $\delta(G(P)) \geqslant 3$, that the faces of P are precisely the faces of $G(P)$, and that every edge of $G(P)$ lies on the boundary of exactly two faces.
- A convex polyhedron P and its corresponding plane graph $G(P)$ are shown in Figure 12.
- Conventionally, V, E, F denote the numbers of vertices, edges and faces of a convex polyhedron P, so Euler's formula can be written

as

$$V - E + F = 2. \tag{5}$$

Equation (5) is the celebrated Euler **polyhedron formula**.

- For convenience, let V_n and F_n denote the numbers of degree-n vertices and degree-n faces of P (equivalently, of $G(P)$).
- Then $n \geqslant 3$, and

$$2E = \sum_{n \geqslant 3} nV_n = \sum_{n \geqslant 3} nF_n. \tag{6}$$

Theorem 3.4.1 Every convex polyhedron has at least one face (polyhedral angle) of degree n with $3 \leqslant n \leqslant 5$.

Proof By contradiction.

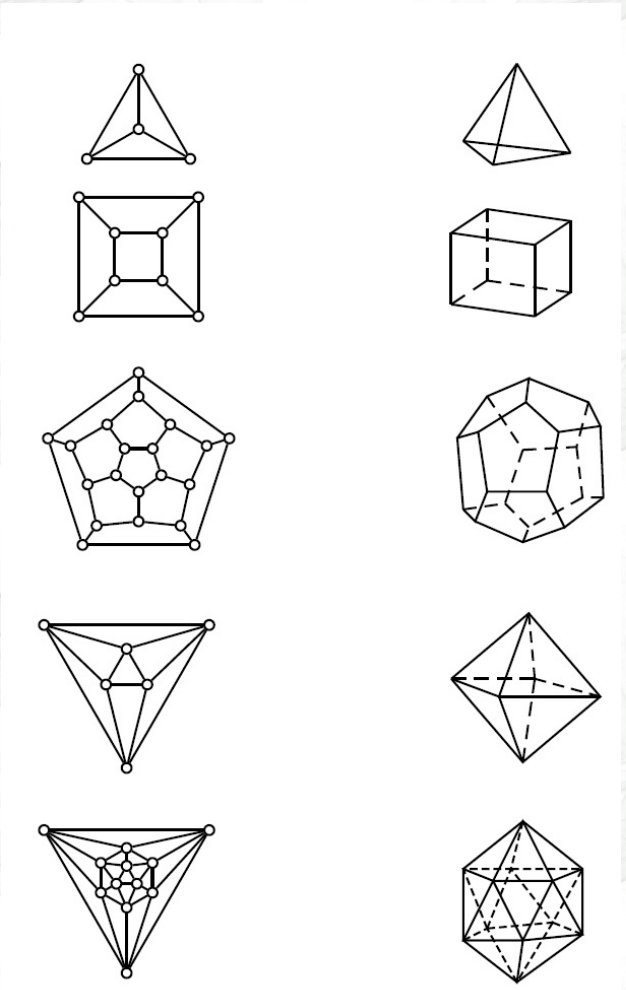


Figure 12: The regular polyhedra

- Suppose $F_3 = F_4 = F_5 = 0$. Then by (6),

$$2E = \sum_{n \geqslant 6} nF_n \geqslant \sum_{n \geqslant 6} 6F_n = 6 \sum_{n \geqslant 6} F_n = 6F,$$

so that

$$F \leqslant \frac{1}{3} E. \tag{7}$$

- On the other hand, again by (6),

$$2E = \sum_{n \geqslant 3} nV_n \geqslant 3 \sum_{n \geqslant 3} V_n = 3V,$$

that is,

$$V \leqslant \frac{2}{3} E. \tag{8}$$

- Combining Euler's polyhedron formula (5) with the inequalities (7) and (8), we obtain

$$E = V + F - 2 \leqslant \frac{2}{3} E + \frac{1}{3} E - 2 = E - 2,$$

a contradiction. This proves the theorem. ∎

A convex polyhedron whose faces are all congruent regular polygons and whose polyhedral angles are all congruent is called a regular polyhedron, also known as a Platonic solid. The corresponding plane graph is called a Platonic graph (see Figure 12).

The following theorem is the last result in the thirteen books of Euclid's *Elements* (ca. 330–275 BC). We prove it using Euler's polyhedron formula.

Theorem 3.4.2 There are exactly five regular polyhedra (shown in Figure 12).

Proof

- Let P be a regular polyhedron and $G(P)$ its corresponding plane

graph. From (5) and (6), for every convex polyhedron we have

$$\begin{aligned} -8 = &\quad 4E-4V-4F = 2E+2E-4V-4F \\ = &\quad \sum_{n\geqslant 3} nF_n + \sum_{n\geqslant 3} nV_n - 4\sum_{n\geqslant 3} V_n - 4\sum_{n\geqslant 3} F_n \\ = &\quad \sum_{n\geqslant 3} (n-4)F_n + \sum_{n\geqslant 3} (n-4)V_n. \end{aligned} \tag{9}$$

- Since P is regular, there are integers $h\,(\geqslant 3)$ and $k\,(\geqslant 3)$ such that $F = F_h$ and $V = V_k$.
- Hence, by (9),

$$-8 = (h-4)F_h + (k-4)V_k \tag{10}$$

and by (6),

$$hF_h = 2E = kV_k. \tag{11}$$

- By Theorem 3.4.1, $3 \leqslant h \leqslant 5, \quad 3 \leq k \leq 5.$

Case 1 When $h = 3$,

- equations (10) and (11) give

$$12 = (6-k)V_k. \tag{12}$$

- From (12) and (10):

  for $k = 3$, $V_3 = 4, F_3 = 4$, so P is the tetrahedron;

  for $k = 4$, $V_4 = 6, F_3 = 8$, so P is the octahedron;

  for $k = 5$, $V_5 = 12, F_3 = 20$, so P is the icosahedron.

Case 2 When $h = 4$,

- equation (10) gives $8 = (4-k)V_k$. Hence $k = 3$, $V_3 = 8, F_4 = 6$, so P is the cube.

Case 3 When $h = 5$,

- equations (10) and (11) give $20 = (10-3k)V_k$. Hence $k = 3$, $V_3 = 20, F_5 = 12$, so P is the dodecahedron. ∎

Exercises

3.4.1 Prove that there is no convex polyhedron having an odd number of faces each of which has an odd number of edges.

3.4.5 Prove that every convex polyhedron has at least one triangular face or one vertex that is a trihedral angle.

5 The Design of Printed Circuit Boards

- When designing and manufacturing printed circuit boards, the first problem one meets is to decide whether a given circuit diagram can be printed on a single layer so that no two wires short-circuit; and, if it can, how to produce a concrete routing. This is called the printed circuit board layout problem.
- Regard the circuit diagram to be printed as an undirected graph (in fact, a simple connected graph) G, whose vertices represent electronic components and whose edges represent wires. The problem then reduces to deciding whether G is planar and, if it is, producing a plane representation of it.
- Unfortunately, Kuratowski's theorem provides no such method. In this section we present the polynomial-time planarity algorithm of Demoucron, Malgrange and Pertuiset (1964), abbreviated as the DMP algorithm; if the graph is planar, the DMP algorithm produces a plane representation of it.

- Before describing the algorithm, the following preprocessing of G greatly reduces the amount of work:

(a) if G is disconnected, test each of its components separately; if G has a cut-vertex, test each of its blocks separately;

(b) deleting loops from G does not affect planarity, and neither does deleting parallel edges;

(c) replacing a degree-2 vertex of G, together with its two incident edges, by a single edge does not affect planarity.

- The last two reductions should be applied repeatedly and alternately until neither can be applied any longer.

- After these reductions, two basic tests can be used for the simple graph G:

(f) if $e(G) < 9$ or $n(G) < 5$, then G must be planar (Exercise 3.2.2);

(g) if $e(G) > 3n-6$ or $\delta(G) > 5$, then by Corollaries 3.1.4.1 and 3.1.4.2, G must be non-planar.

- Let G be a planar graph, $H \subseteq G$, and let $\widetilde{H}$ be a plane representation of H. If there is a plane representation $\widetilde{G}$ of G such that $\widetilde{H} \subseteq \widetilde{G}$, then $\widetilde{H}$ is called G-admissible.

- For example, Figure 13(a) shows a planar graph G, while (b) and (c) show two plane representations of $G - xy$; the one in (b) is G-admissible, whereas the one in (c) is not.

- Let B be an H-bridge of G (for the definition of an H-bridge see Section 3.2), and let f be a face of $\widetilde{H}$. If the set of vertices of attachment of B to H in G satisfies $V_G(B,H) \subseteq B_{\widetilde{H}}(f)$, then B is said to be drawable in the face f.

- Let $F_G(B,\widetilde{H})$ denote the set of those faces f of $\widetilde{H}$ in which B is drawable, that is, $F_G(B,\widetilde{H}) = \{f \in F(\widetilde{H}) : B \text{ is drawable in } f\}$.

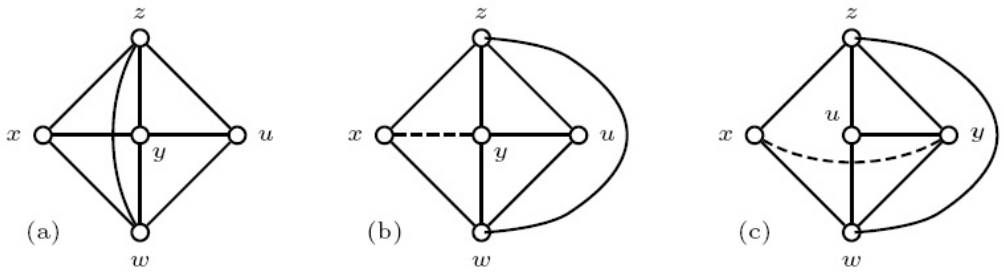


Figure 13: (a) A planar graph G; (b) a G-admissible subgraph G_1; (c) a non-G-admissible subgraph G_2

The following theorem gives a necessary condition for G to be planar.

Theorem 3.5.1 If $\widetilde{H}$ is G-admissible, then for every H-bridge B of G, $F_G(B,\widetilde{H}) \neq \emptyset$.

Proof

- Suppose $\widetilde{H}$ is G-admissible. By definition, there is a plane representation $\widetilde{G}$ of G with $\widetilde{H} \subseteq \widetilde{G}$.

- By the definition of an H-bridge, the subgraph of $\widetilde{G}$ corresponding to B must be confined to a single face of $\widetilde{H}$.
- Therefore, $F_G(B, \widetilde{H}) \neq \emptyset$. ∎

The planarity algorithm

The DMP planarity algorithm:

1. Let G_1 be a cycle of G (we may assume that G is connected and has no cut-vertex), and find a plane representation $\widetilde{G}_1$ of G_1. Set $i = 1$.

2. If $E(G) \setminus E(G_i) = \emptyset$, stop. Otherwise, determine all the G_i-bridges of G, and for each G_i-bridge B find the set $F_G(B, \widetilde{G}_i)$.

3. If there is a G_i-bridge B with $F_G(B, \widetilde{G}_i) = \emptyset$, stop; by Theorems 3.5.1 and 3.5.2, G is non-planar. If there is a G_i-bridge B with $|F_G(B, \widetilde{G}_i)| = 1$, let $\{f\} = F_G(B, \widetilde{G}_i)$; if no such bridge exists, let B be any G_i-bridge and choose any $f \in F_G(B, \widetilde{G}_i)$.

4. Choose an xy-path $P_i \subseteq B$ with $x, y \in V_G(B, G_i)$. Set $G_{i+1} = G_i \cup P_i$, and draw P_i inside the face f of $\widetilde{G}_i$ to obtain a plane representation $\widetilde{G}_{i+1}$ of G_{i+1}. Replace i by $i+1$ and return to Step 2.

Example 3.5.1 To illustrate the algorithm, consider the graph G shown in Figure 14. Start with the cycle $\widetilde{G}_1 = (2345672)$ (for simplicity, each bridge is represented by its edge set). At each step, the bridge B for which $|F_G(B, \widetilde{G}_i)| = 1$ is marked in bold. The algorithm terminates with a plane representation $\widetilde{G}_{11}$ of G, so G is planar. ∎

Theorem 3.5.2 The DMP planarity algorithm is valid.

Proof

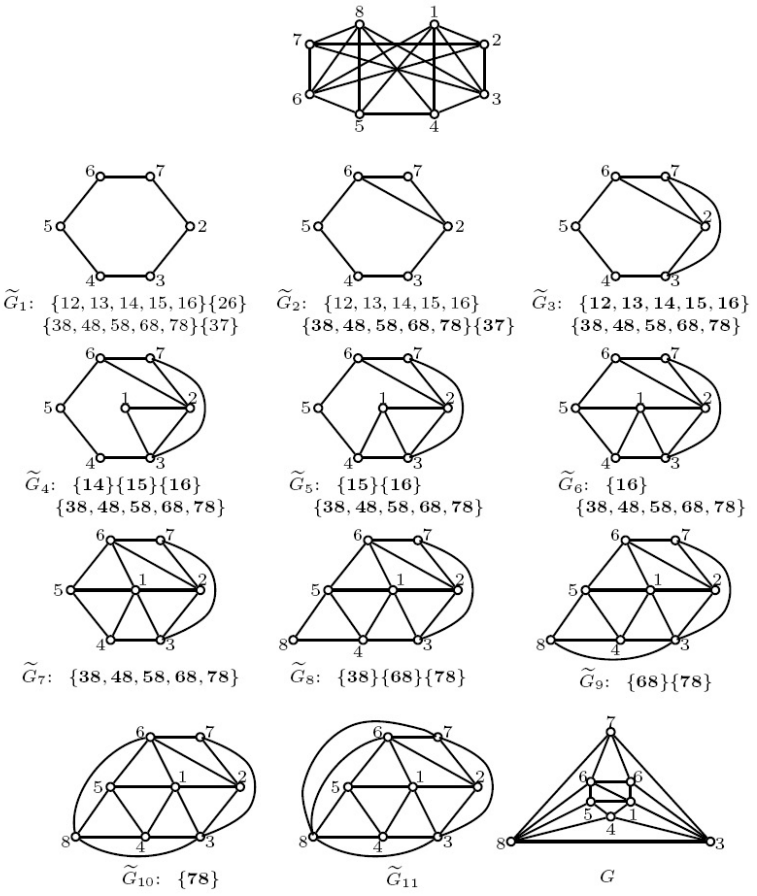


Figure 14: An application of the DMP planarity algorithm

- If G is a connected planar graph, then $\phi(G) = e(G) - n(G) + 2$.
- Put $m = e(G)$ and $n = n(G)$.
- The DMP algorithm starts from a cycle G_1 of G, so $\phi(G_1) = 2$.
- During the algorithm, whenever i is replaced by $i+1$, the number of faces increases by one, that is, $\phi(G_{i+1}) = \phi(G_i) + 1 = i + 2$.
- Hence, if G is planar, the algorithm must terminate after producing the sequence $\widetilde{G}_1, \cdots, \widetilde{G}_{m-n+1}$.
- It therefore suffices to prove that if G is planar, then each of $\widetilde{G}_1, \cdots, \widetilde{G}_{m-n+1}$ is G-admissible.
- We induct on $k = 1, 2, \ldots, m-n+1$.
- For $k = 1$, since G is planar, $\widetilde{G}_1$ is clearly G-admissible.
- Assume that for every i $(1 \leqslant i \leqslant k < m-n+1)$, $\widetilde{G}_i$ is G-admissible. By definition, there is a plane representation $\widetilde{G}$ of G with $\widetilde{G}_k \subseteq \widetilde{G}$. We now show that $\widetilde{G}_{k+1}$ is G-admissible.

- Let B and f be the G_k-bridge and the face of $F_G(B, \widetilde{G_k})$ chosen at the k-th step.
- If $F_G(B, \widetilde{G}_k) = \{f\}$, then the representation $\widetilde{G}_{k+1}$ constructed by the algorithm clearly satisfies $\widetilde{G}_k \subseteq \widetilde{G}_{k+1} \subseteq \widetilde{G}$.
- So suppose instead that $|F_G(B', \widetilde{G}_k)| > 1$ for every G_k-bridge B' of G, and assume that B is drawn in another face f' of $F_G(B, \widetilde{G}_k)$.
- Since G is connected and has no cut-vertex, every G_k-bridge B' satisfies $|V_G(B', G_k)| \geqslant 2$.
- Thus every G_k-bridge B' whose vertices of attachment $V_G(B', G_k)$ lie on the common boundary of f and f' can be drawn equally well in f or in f'.
- For such a bridge B', obtain another plane representation $\widetilde{G}'$ of G as follows: if B' is drawn in the face f' of $\widetilde{G}$, draw it in f in $\widetilde{G}'$; if B' is drawn in f in $\widetilde{G}$, draw it in f'.

- (For instance, see Figure 15: the bridge B and the face f are those chosen at the third step, but B is drawn in the face f'. Then the two bridges B and B_2 in f' are drawn in f, while the bridge B_1 in f is drawn in f'.)
- Therefore $\widetilde{G}_{k+1} \subseteq \widetilde{G}'$, i.e. $\widetilde{G}_{k+1}$ is G-admissible. ∎

Example 3.5.2 Consider the graph H in Figure 16. Start with the cycle $\widetilde{H}_1 = (123487651)$ and proceed in the order shown in the figure. In $\widetilde{H}_5$ there is an $\widetilde{H}_5$-bridge $B = \{36\}$ with $F_H(B, \widetilde{H}_5) = \emptyset$. The algorithm stops, and by Theorem 3.5.1, H is non-planar. ∎

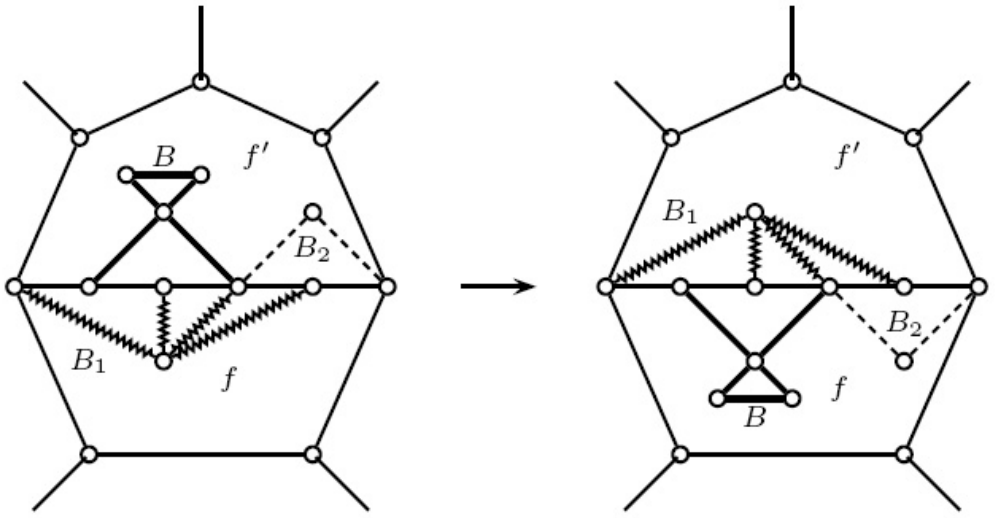


Figure 15: Auxiliary figure for the proof of Theorem 3.5.2

Exercises

None.

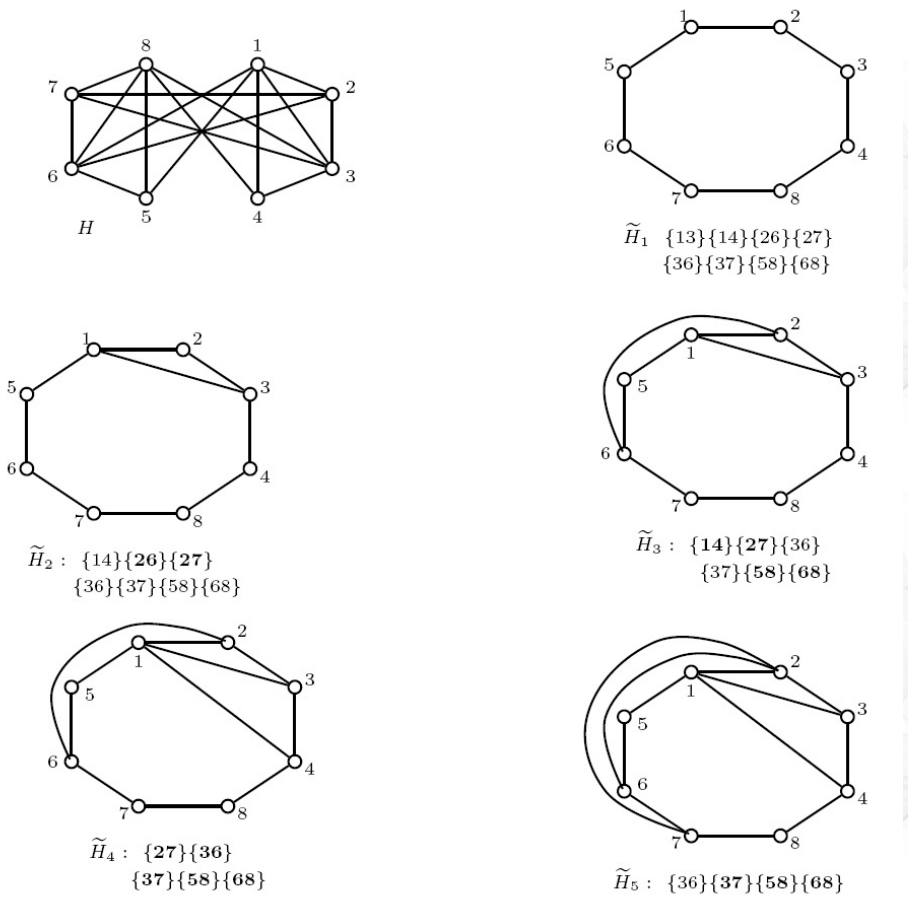


Figure 16: An application of the DMP planarity algorithm

Thank you!