Exercise 1. Prove: let D be a simple graph, and $k = \max\{\delta^+, \delta^-\}$, then D contains a directed path of length at least k.

pf:

Let $P = (x_0, x_1, \cdots, x_l)$ be the longest path in D.

Then $N_D^-(x_0) \subset \{x_j\}_{j=0}^{l}$,

otherwise $\exists\ \tilde{x} \in N_D^-(x_0)$, $\tilde{P} = (\tilde{x}, x_0, \cdots, x_l)$

$|E(\tilde{P})| > |E(P)|$, a CONTRADICTION.

Similarly, $N_D^+(x_l) \subset \{x_j\}_{j=0}^{l}$.

Hence $l \geq |N_D^-(x_0)| \geq \delta^-$

$l \geq |N_D^+(x_k)| \geq \delta^+$

Above all, $l \geq \max\{\delta^-, \delta^+\} = k$.

□

Exercise 2. Let G be a simple graph of order n, and $\omega = \omega(G)$. Prove:

(a) $e(G) \leq \frac{1}{2}(n-\omega)(n-\omega+1)$;

(b) If $e(G) > \frac{1}{2}(n-1)(n-2)$, then G is connected.

(a) pf: Let $\{V_j\}_{j=1}^{w}$ be the vertices set of connected components, V_j be the order of each V_j.

Thus $e(G) \leq \sum_{j=1}^{w} \frac{1}{2} V_j(V_j - 1)$.

$$\sum_{j=1}^{w} v_j = n.$$

Let $M = \frac{1}{2}\sum_{j=1}^{w} v_j^2 - \frac{1}{2}n$

Assum v_1 be the max one.

Consider $\tilde{v}_1 = v_1 + 1$

$\tilde{v}_k = v_k - 1 \quad (k \neq 1)$

Otherwise $\tilde{v}_j = v_j$.

$$\tilde{M} - M = \frac{1}{2}[(v_1+1)^2 + (v_k-1)^2 - v_1^2 - v_k^2]$$

$$= \frac{1}{2}[2v_1 - 2v_k + 2]$$

$$= v_1 - v_k + 1 > 0.$$

Hence $M \leq \frac{1}{2}(n-w+1)(n-w+1) + \frac{1}{2}(w-1) - \frac{1}{2}n$

$$= \frac{1}{2}(n-w+1)(n-w)$$

$$e(G) \leq \frac{1}{2}(n-w+1)(n-w).$$

The equality holds iff $G[V_1] = K_{n-w+1}$, $G[V_j] = K_1$, $j \neq 1$.

□.

(b) Assume $w \geq 2$,

Thus $e(G) \leq \frac{1}{2}(n-1)(n-2)$, a CONTRADICTION.

Hence $w=1$, G is connected.

Exercise 3. Let G be a non-bipartite simple graph of order n, $k \geq 2$, and

$$\delta(G) > \left\lfloor \frac{2n}{2k+1} \right\rfloor.$$

Prove that G contains an odd cycle of length $\leq 2k-1$.

Pf: By König theorem, there exists an odd cycle in G.

Let C be the minimum odd cycle.

Assum $|E(C)| \geq 2k+1$.

① $\forall x, y \in V(C)$, $xy \notin E(G)$.

If not so, then x, y divide C into two parts, one of the length is odd, and the other is even.

Thus we can find an odd cycle is shorter than C, a CONTRADICTION.

② $\forall x \in V(G-C)$, $|N_G(x) \cap C| \leq 2$.

If not so, let $N_G(x) \cap C = \{y_1, y_2, y_3, \cdots\}$.

Then y_1, y_2, y_3 divide C into three parts, and there exists one part whose length is odd.

Thus we find an odd cycle is shorter than C, a CONTRADICTION.

Now we consider double counting the sum of degree of $V(C)$.

Let $|V(C)| = s$. We know $s \geq 2k+1$ by assumption.

$$\sum_{x \in V(C)} d_G(x) = 2s + E(V(C), V(G-C))$$
$$\leq 2s + 2(n-s)$$
$$= 2n$$

Hence $2n \geq s \cdot \delta(G)$
$$\geq (2k+1) \cdot \delta(G).$$

Based on the condiction given in the problem,

$$\delta(G) > \left\lfloor \frac{2n}{2k+1} \right\rfloor.$$

Since $\delta(G)$ is an integer,

$$\delta(G) > \frac{2n}{2k+1}, \quad (2k+1) \cdot \delta(G) > 2n.$$

A CONTRADICTION.

Hence $E(C) \leq 2k-1$,

there exists an odd cycle whose length is less than $2k-1$.

□.