Exercise 1. Prove that there is no convex polyhedron having an odd number of faces each of which has an odd number of edges.

Pf: 凸多面体可嵌入到平面. 记其平面表示为 G.

由平面图的握手定理:

$$\sum_{f\in F(G)} d_G(f) = 2|E(G)|.$$

记 $F_{odd}(G) = \{ f \in G : d_G(f) \equiv 1 \pmod 2 \}$.

$F_{even}(G) = \{ f \in G : d_G(f) \equiv 0 \pmod 2 \}$.

故
$$\sum_{f\in F(G)} d_G(f) = \sum_{f\in F_{odd}(G)} d_G(f) + \sum_{f\in F_{even}(G)} d_G(f)$$

$$\equiv \sum_{f\in F_{odd}(G)} d_G(f)$$

$$\equiv 0 \pmod 2.$$

故不存在凸多面体: 它有奇数个面, 而且每个面有奇数条棱.

□.

Exercise 2. Prove that every convex polyhedron has at least one triangular face or one vertex that is a trihedral angle.

Pf: 取凸多面体的平面表示 G.

假设任取点 v, $d_G(v) \geq 4$.

任取面 f, $d_G(f) \geq 4$.

则
$$\sum_{v\in V(G)} d_G(v) = 2m \geq 4n$$

$$\sum_{f\in F(G)} d_G(f) = 2m \geq 4\phi.$$

其中，$n = |V(G)|$，$m = |E(G)|$，$\phi = |F(G)|$.

故 $2 = n - m + \phi = \frac{1}{3}m - m + \frac{2}{3}m = 0$，矛盾.

故假设成立，即存在一个3度面或一个3度点.

□.

Exercise 3. Suppose that $N = (D_{xy}, \mathbf{c})$ is a capacity network and $B = (S, \overline{S})$ is an (x, y)-edge cut. Prove that for every (x, y)-flow $\mathbf{f}$ in N,

$$\mathbf{f}(B) - \mathbf{f}^-(S) = \mathbf{f}^+(S) - \mathbf{f}^-(S) = \begin{cases} \operatorname{val}\mathbf{f}, & \text{if } x \in S; \\ -\operatorname{val}\mathbf{f}, & \text{if } y \in S. \end{cases}$$

pf: 如果 $x \in S$，

$$\sum_{v \in S} (f^+(v) - f^-(v)) = f^+(x) - f^-(x) = \operatorname{val} f$$

如果 $y \in S$，

$$\sum_{v \in S} (f^+(v) - f^-(v)) = f^+(y) - f^-(y) = -\operatorname{val} f$$

□.

Exercise 4. Let $N = (D_{xy}, \mathbf{c})$ be a capacity network, let $\mathbf{f}$ be an (x, y)-flow in N, and let $B = (S, \overline{S})$ be an (x, y)-edge cut. Prove that

(a) if $\mathbf{f}(a) = \mathbf{c}(a)$ for every $a \in (S, \overline{S})$ and $\mathbf{f}(a) = 0$ for every $a \in (\overline{S}, S)$, then $\mathbf{f}$ is a maximum flow and B is a minimum cut;

(b) if $\mathbf{f}$ is a maximum flow and B is a minimum cut, then $\mathbf{f}(a) = \mathbf{c}(a)$ for every $a \in (S, \overline{S})$ and $\mathbf{f}(a) = 0$ for every $a \in (\overline{S}, S)$.

pf: (a)

记流 f 的残量网络 $(D', \tilde{f})$，

$\tilde{f}$ 的支撑图为 $\tilde{D}$.

$\forall a \in B = (S, \overline{S})$，$\tilde{f}(a) = c(a) - f(a) = 0$

$\forall a \in (\overline{S}, S)$，记 a 在 D' 的反向弧为 $\tilde{a}$.

$\tilde{f}(\bar{a}) = f(a) = 0$

故 $\forall a' \in E_{D'}(S, \bar{S})$, $\tilde{f}(a') = 0$.

在 D^* 中, $E_{D^*}(S, \bar{S}) = \phi$.

故没有 $x-y$ 增广路.

取 $R = \{v$: 存在有向路从 x 到 $v\}$. $C = C(R, \bar{R})$.

同样 $E_{D^*}(R, \bar{R}) = \phi$.

$\forall a \in C(R, \bar{R})$, $\tilde{f}(a) = 0$, 故 $f(a) = c(a)$

$\forall a \in (\bar{R}, R)$, $\tilde{f}(\bar{a}) = 0$, 故 $f(a) = 0$

$$\begin{aligned} val f &= f^+(R) - f^-(R) = cap R - 0 \\ &= cap R \end{aligned}$$

$\forall f'$ 为一个 (x, y) 流, $val f = cap R \geq f'^+(R) - f'^-(R) = val f'$.

f 为最大流.

$$\begin{aligned} val f &= f^+(S) - f^-(S) \\ &= cap B - 0 \\ &= cap B \end{aligned}$$

$\forall B'$ 为一个 (x, y)-edge cut $(S', \bar{S}')$

$$\begin{aligned} val f &= f^+(S') - f^-(S') \\ &\leq f^+(S') \leq cap B' \end{aligned}$$

故 $\forall B'$, $cap B' \geq cap B$.

故 B 为最小割.

(b) $$\begin{aligned} val f &= f^+(S) - f^-(S) \\ &\leq f^+(S) \quad \cdots ① \\ &\leq c^+(S) \quad \cdots ② \\ &= cap B \end{aligned}$$

由最大流最小割定理，$val f = cap B$.

故①②式等号成立.

即 $\forall a \in (\bar{S}, S)$, $f(a) = 0$.

$\forall a \in (S, \bar{S})$, $f(a) = c(a)$.

□.