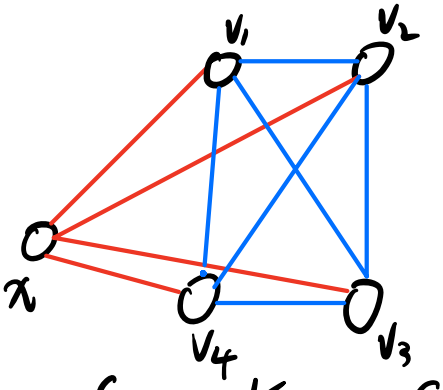
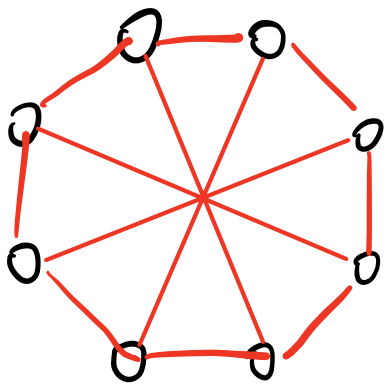
Exercise 1. Prove that $R(3,4) = 9$.

Pf:



Let $G = K_9$. Considering the color setting. Hence $v(G) = 9$, at least 4 red or blue edges are incident to x. Fix the above color to red. Let v_1, v_2, v_3, v_4 denote the vertices connected to x by red edges. Then $G[v_1, \cdots, v_4]$ must be a blue K_4, or red K_3 occurs. Above all, $R(3,4) \leq 9$.



When $n = 8$, we can construct a counterexample. As shown above, the unconnected edges are all blue edges in K_8. We can see there's no red K_3 or blue K_4 in it.

In summary, $R(3,4) = 9$.

□.

Exercise 2. Nine mathematicians meet at an international conference. Among any three of them, at least two speak the same language. Prove: if each mathematician speaks at most three languages, then at least three mathematicians can converse in the same language.
(7th US Mathematical Olympiad problem, May 2, 1978.)

Pf:
Let G be a graph on 9 people.
Two vertices are adjacent if the corresponding people speak a common language.
Assume that no 3 people speak a common language. Then each language is spoken by at most 2 people.
Each person speak at most 3 language.
Therefore, each vertex of G has degree at most 3.
By the handshaking lemma,

$$2e(G) = \sum_{v \in V(G)} d_G(v) \leq 3 \times 9 = 27.$$

$$e(G) \leq 13.$$

Consider $\overline{G}$. (complement of G)
Among any 3 people, at least two speak a common language. So there's no triangle in $\overline{G}$.
By Mantel's theorem,

$$e(\overline{G}) \leq \left\lfloor \frac{9^2}{4} \right\rfloor = 20.$$

On the other hand,

$$e(G) + e(\overline{G}) = e(K_9) = \frac{9 \times 8}{2} = 36$$

A contradiction.

So there exists 3 people speak a common language.

□.

Exercise 3. Prove that $\mathrm{Aut}(C_n) = D_n$, where D_n denotes the dihedral group.

Each rotation and reflection is in $Aut(C_n)$.

So $D_n < Aut(C_n)$.

Consider $\varphi \in Aut(C_n)$.

Assume $V(C_n) = \{v_0, \dots, v_{n-1}\}$, $E(C_n) = \{\{v_j, v_{j+1}\}\}_{j=0}^{n-1}$

Subscripts are module by n.

There's n choices for $\varphi(v_0)$.

After v_0 is chosen, there's 2 choices for $\varphi(v_1)$.

Then $v_2, \dots, v_{n-1}$'s image is determined.

Hence $|Aut(C_n)| \leq 2n$.

Providing that $|D_n| = 2n$,

$Aut(C_n) = D_n$.

Exercise 4. Prove: Let G be a simple undirected graph with vertex set $V = \{x_1, x_2, \dots, x_n\}$ containing no subgraph K_{r+1}. Then there exists a complete r-partite graph H with vertex set V such that for every i $(1 \leq i \leq n)$,

$$d_G(x_i) \leq d_H(x_i),$$

with equality if and only if $G \cong H$.
(P. Erdős, 1970)

pf:

We prove the statement by induction on r.

- For $r = 1$. G contains no subgraph K_2.

Thus G has no edges.

Choosing $H = G$ is ok.

- Suppose $r \geq 2$. Assume the result is true for $r-1$.
Choose a vertex v of maxium degree in G.
Let $A = N_G(v)$, $B = V(G) \setminus A$.
Thus $|A| = d_G(v) = \Delta(G)$.
$G[A]$ contains no K_r. If not so,
$G[V(K_r) \cup \{v\}] = K_{r+1}$. A contradiction.
By the induction hypothesis,
there's a complete $(r-1)$-partite graph H_A on A
s.t. $d_{G[A]}(x) \leq d_{H_A}(x)$, $\forall x \in A$.
Now we construct H.
Use the $r-1$ parts of H_A as the first $r-1$ parts of H. Use B as the last part. Join every vertex of A to every vertex of B. Thus H is a complete r-partite graph.
Let $x \in A$. Then

$$\begin{aligned} d_G(x) &= |N_G(x)| \\ &= |N_G(x) \cap A| + |N_G(x) \cap B| \\ &= d_{G[A]}(x) + |N_G(x) \cap B| \\ &\leq d_{G[A]}(x) + |B| \quad \cdots ① \\ &\leq d_{H_A}(x) + |B| = d_H(x). \quad \cdots ② \end{aligned}$$

Let $x \in B$. Then

$$d_G(x) \leq \Delta(G) = d_G(v) = |A|$$
$$d_H(x) = |A|.$$

Hence $d_G(x) \leq d_H(x)$, $\forall x \in V(G)$.

Now we consider the "=" case.
Suppose that $d_G(x) = d_H(x)$, $\forall x \in V(G)$.
$\forall x \in A$, "=" must hold on ① & ②.
Thus $d_{G[A]}(x) = d_{H_A}(x)$.
$|N_G(x) \cap B| = |B|$. => every vertex of A is adjacent to every vertex of B.
By induction hypothesis,
$G[A] \cong H_A$.

$\forall y \in B$, $d_G(y) = d_H(y) = |A|$.
y is adjacent to every vertex of A.
Thus y has no neighbor in B.
So B is an independent set.
It follows that G is a complete r-partite graph with the same size as H.
Hence, $G \cong H$.
Conversely, suppose $G \cong H$,
then $d_G(x) = d_H(x)$, $\forall x \in V(G)$.

□