**Exercise 1.** Let $T$ be a nontrivial tree. Prove:

(a) $T$ is bipartite;

(b) Let $\{X, Y\}$ be the bipartition of $T$. Then:

(i) if $|X| \geq |Y|$, then $X$ contains at least one vertex of degree 1;

(ii) if $|X| = |Y| + k$, then $X$ contains at least $k+1$ vertices of degree 1.

Pf: (a) $T$ contains no odd cycle.
According to the biparitite graph characterization theoren, $T$ is bipartite

(b) Let $X_1 = \{x \in X : d_T(x) = 1\}$.

Then $\sum_{x \in X} d_T(x) \geq |X_1| + 2(|X| - |X_1|)$

$= 2|X| - |X_1|$.

Since $T$ is a tree,

$\sum_{x \in X} d_T(x) = e(T) = |X| + |Y| - 1$

Hence, $|X_1| \geq |X| - |Y| + 1$.

Therefore, (i), (ii) is true.

□.

**Exercise 2.** Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a family of $n$ distinct subsets of $X = \{1, 2, \ldots, n\}$. Prove that there exists $x \in X$ such that $A_1 \cup \{x\}, A_2 \cup \{x\}, \ldots, A_n \cup \{x\}$ are all distinct.

pf: Let $\mathcal{A}^c = \{X \setminus A_1, X \setminus A_2, \cdots, X \setminus A_n\}$.

$X \setminus A_j \neq X \setminus A_k$, $j \neq k$

By Bondy's Theorem, there exists $x \in X$, $\{(X \setminus A_j) \setminus \{x\}\}_{j=1}^n$ are all distinct.

Hence, $\{X \setminus (A_j \cup \{x\})\}_{j=1}^n$ are all distinct.

Similarly, $A_1 \cup \{x\}, A_2 \cup \{x\}, \ldots, A_n \cup \{x\}$ are all distinct.

**Exercise 3.** Prove: if $B_1$ and $B_2$ are bonds, then their symmetric difference $B_1 \triangle B_2$ is an edge-cut set and hence contains a bond.

pf: Let $B_1 = E_G(S_1, \bar{S}_1)$

$B_2 = E_G(S_2, \bar{S}_2)$.

We prove $B_1 \triangle B_2 = E_G(S_1 \triangle S_2, \overline{S_1 \triangle S_2})$ as follows.

(C): $\forall\, e = xy \in B_1 \triangle B_2$,

$x \in S_1 \triangle S_2$,

$y \in \bar{S}_1 \triangle \bar{S}_2$

$= \underline{\bar{S}_1 \cup \bar{S}_2} \,/\, (\bar{S}_1 \cap \bar{S}_2)$

$= \underline{S_1 \cap S_2} \,/\, \overline{S_1 \cup S_2}$

$= \overline{S_1 \cup S_2 \,/\, S_1 \cap S_2}$

$= \overline{S_1 \triangle S_2}$.

$xy \in E_G(S_1 \triangle S_2, \overline{S_1 \triangle S_2})$.

$B_1 \triangle B_2 \subset E_G(S_1 \triangle S_2, \overline{S_1 \triangle S_2})$.

(⊃): Similarly, it's correct.

Thus $B_1 \triangle B_2$ is an edge-cut set, containing a bond.

**Exercise 4.** Let $C$ be a cycle and $B$ a bond of $D$. Prove:

(a) $\mathbf{f}_C \in \mathcal{E}(D)$ is a cycle vector of $D$, and all cycle vectors of $D$ form a subspace of $\mathcal{E}(D)$;

(b) $\mathbf{g}_B \in \mathcal{E}(D)$ is a cut vector of $D$, and all cut vectors of $D$ form a subspace of $\mathcal{E}(D)$.

(a)

pf: Let $x \in V(C)$, $e_1, e_2$ be the two edges of $C$ that enter or leave $x$.

1. If the directions of both edges agree with the orientation of $C$, then
$$f_C^+(x) = f_C^-(x) = 1.$$

2. If only the orientation of $e_1$ is reversed, then
$$f_C^+(x) = 1 + (-1) = 0 = f_C^-(x)$$

3. If only the orientation of $e_2$ is reversed, then
$$f_C^+(x) = 0 = 1 + (-1) = f_C^-(x)$$

4. If all are reversed, then
$$f_C^+(x) = f_C^-(x) = -1.$$

$\forall y \in V(D) \setminus V(C)$, $f_C^+(y) = f_C^-(y) = 0$.

Thus, $f_C$ is a cycle vector.

Next we prove $\mathbb{C}(D)$ is a subspace of $\mathbb{E}(D)$.

① Obviously, $0 \in \mathbb{C}(D)$

② $\forall f, g \in \mathbb{C}(D)$, $x \in V(D)$,
$$f^+(x) = f^-(x), \quad g^+(x) = g^-(x).$$
Hence, $(f+g)^+(x) = (f+g)^-(x)$.
$f+g \in \mathbb{C}(D)$.

③ $\forall f \in \mathbb{C}(D)$, $\alpha \in \mathbb{R}$.

$(\alpha f)^+(x) = \alpha f^+(x) = \alpha f^-(x) = (\alpha f)^-(x)$

$\alpha f \in \mathbb{C}(D)$.

□.

(b) Let $p(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S. \end{cases}$

1. If $xy \in E_D^+(S, \bar{S})$, $g_B(xy) = 1 = p(x) - p(y)$
2. If $xy \in E_D^-(S, \bar{S})$, $g_B(xy) = -1 = p(x) - p(y)$
3. If $xy \notin B$, $g_B(xy) = 0 = p(x) - p(y)$.

Hence $g_B = \delta_p \in \mathbb{B}(D)$.

Next, we prove $\mathbb{B}(D)$ is a subspace of $\mathbb{E}(D)$.

① Obviously, $0 \in \mathbb{B}(D)$.

② $\forall \delta_{p_1}, \delta_{p_2} \in \mathbb{B}(D)$, $xy \in E(D)$.

$$\begin{aligned} &(\delta_{p_1} + \delta_{p_2})(xy) \\ &= \delta_{p_1}(xy) + \delta_{p_2}(xy) \\ &= p_1(x) - p_1(y) + p_2(x) - p_2(y) \\ &= (p_1 + p_2)(x) - (p_1 + p_2)(y) \\ &= \delta_{p_1+p_2}(xy). \end{aligned}$$

$\delta_{p_1} + \delta_{p_2} \in \mathbb{B}(D)$.

③ $\forall \delta_p \in \mathbb{B}(D)$, $\alpha \in \mathbb{R}$, $xy \in E(D)$

$$\begin{aligned} (\alpha\delta_p)(xy) &= \alpha\, \delta_p(xy) = \alpha(p(x) - p(y)) \\ &= \delta_{\alpha p}(xy). \end{aligned}$$

$\alpha\delta_p \in \mathbb{B}(D)$.

**Exercise 5.** Let $\mathbf{K}$ be obtained from the incidence matrix $\mathbf{M}$ of a connected graph $D$ by deleting any one row. Prove that $\mathbf{K}$ is a basis matrix of the cut space $\mathcal{B}(D)$.

pf: We have $\mathcal{B}(D) = \text{Span}\{M_x\}_{x \in V(D)}$.

$\dim \mathcal{B}(D) = n - w(D) = n - 1$.

By the definition of $M$,

$$\sum_{x \in V(D)} M_x(a) = 0, \quad \forall a \in E(D)$$

Hence $\sum_{x \in V(D)} M_x = 0$.

If we delete the row of $x_0$,

then we assert that

$\text{Span}\{M_x\}_{x \in V(D) \setminus x_0} = \text{Span}\{M_x\}_{x \in V(D)}$.

"$\subset$" is obvious, now we consider "$\supset$".

$$\sum_{x \in V(D)} \alpha_x M_x = \sum_{x \in V(D) \setminus x_0} \alpha_x M_x + \alpha_{x_0} M_{x_0}$$

$$= \sum_{x \in V(D) \setminus x_0} \alpha_x M_x + \alpha_{x_0} \left( \sum_{x \in V(D)} M_x - \sum_{x \in V(D) \setminus x_0} M_x \right)$$

$$= \sum_{x \in V(D) \setminus x_0} (\alpha_x - \alpha_{x_0}) M_x.$$

Now we have proved the assertion.

Since $\dim \text{span}\{M_x\}_{x \in V(D) \setminus x_0} = n - 1$,

$\mathbf{K}$ is a basis matrix of the cut space $\mathcal{B}(D)$.